30. **D-modules - II**

We would now like to explain how the Beilinson-Bernstein localization theorem can be used to classify various kinds of irreducible representations of $\mathfrak{g}$. For this we will need to build up a bit more background on D-modules.

30.1. **Support of a quasicoherent sheaf.** Let M be a quasicoherent sheaf on a variety X, and $Z \subset X$ a closed subvariety. We will say that M is **supported** on Z if for any affine open set $U \subset X$, regular function $f \in \mathcal{O}(U)$ vanishing on Z, and $v \in M(U)$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $f^N v = 0$. The **support** $\mathrm{Supp}(M)$ is then defined as the intersection of all closed subvarietes $Z \subset X$ such that M is supported on Z. So M is supported on Z iff the support of M is contained in Z.

In particular, we can talk about support of a (left or right, possibly twisted) D-module on a smooth variety X. The category of D-modules on X supported on Z will be denoted by $\mathcal{M}_Z(X)$.

Example 30.1. It is easy to see that $\mathbb{C}[x, x^{-1}]$ is a left D-module on $\mathbb{A}^1$, and $\mathbb{C}[x]$ is its submodule. These modules have full support $\mathbb{A}^1$. On the other hand, consider the quotient $\delta_0 := \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$.[21] It is clear that δ_0 has a basis $v_i = x^{-i}$, $i \geq 1$, with $xv_i = v_{i-1}$, $xv_1 = 0$, $\partial v_i = -iv_{i+1}$. Thus the support of δ_0 is $\{0\}$.

30.2. **Restriction to an open subset.** Recall that if $\mathcal{A}$ is an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory (i.e., a full subcategory closed under taking subquotients and extensions) then one can form the quotient category $\mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$, but with $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ being the direct limit of $\mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$ over $X' \subset X$ and $Y' \subset Y$ such that $X', Y' \in \mathcal{B}$. One can show that $\mathcal{A}/\mathcal{B}$ is an abelian category. The natural functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is then called the **Serre quotient functor**. This functor is essentially surjective, its kernel is $\mathcal{B}$, and it maps simple objects to simple objects or zero. Thus F defines a bijection between simple objects of $\mathcal{A}$ not contained in $\mathcal{B}$ and simple objects of $\mathcal{A}/\mathcal{B}$.

For example, if X is a variety, $Z \subset X$ a closed subvariety, $\mathrm{Qcoh}(X)$ the category of quasicoherent sheaves on X and $\mathrm{Qcoh}_Z(X)$ the full subcategory of sheaves supported on Z then $\mathrm{Qcoh}(X)/\mathrm{Qcoh}_Z(X) \cong \mathrm{Qcoh}(X \setminus Z)$. The corresponding Serre quotient functor is the restriction $M \mapsto M|_{X \setminus Z}$.

[21]In analysis δ_0 arises as the D-module generated by the δ-function at zero, which motivates the notation.

Now assume that X is smooth. Let $j : X \setminus Z \hookrightarrow X$ be the open embedding. Then we have a **restriction functor** on D-modules

$$j^! : \mathcal{M}(X) \to \mathcal{M}(X \setminus Z)$$

which is the usual restriction functor at the level of sheaves; it is also called the **inverse image** or **pull-back** functor, since it is a special case of the inverse image functor defined above. Thus $j^!(M) = 0$ if and only if M is supported on Z and the functor $j^!$ is a Serre quotient functor which induces an equivalence $\mathcal{M}(X)/\mathcal{M}_Z(X) \cong \mathcal{M}(X \setminus Z)$.

The functor $j^!$ has a right adjoint **direct image (or push-forward) functor**

$$j_* : \mathcal{M}(X \setminus Z) \to \mathcal{M}(X),$$

which is just the sheaf-theoretic direct image (=push-forward). Namely, for an affine open $U \subset X$, $j_*M(U) := M(U \setminus Z)$ regarded as a module over $D(U) \subset D(U \setminus Z)$. While the functor $j^!$ is exact, the functor j_* is only left exact, in general (as so is the push-forward functor for sheaves). In particular, j_* is **not** the (right exact) direct image defined above since the morphism j is not affine, in general; rather it is the zeroth cohomology of the full direct image functor defined on the derived category of D-modules, which we will not discuss here. They do agree, however, when j is affine (e.g., when Z is a hypersurface).

In particular, $j^!$ defines a bijection between isomorphism classes of simple D_X-modules which are not supported on Z and simple $D_{X\setminus Z}$-modules, given by $M \mapsto j^!M$.

The inverse map is defined as follows. Given $L \in \mathcal{M}(X\setminus Z)$, consider the D-module j_*L. Since j_* is right adjoint to $j^!$, the module j_*L does not contain nonzero submodules supported on Z. Now define $j_{!*}L$ to be the intersection of all submodules N of j_*L such that j_*L/N is supported on Z. This gives rise to a functor $j_{!*} : \mathcal{M}(X \setminus Z) \to \mathcal{M}(X)$ (not left or right exact in general). Then if L is irreducible, so is $j_{!*}L$, and $j^!j_{!*}L \cong L$, while for $M \in \mathcal{M}(X)$ irreducible and not supported on Z we have $j_{!*}j^!M \cong M$. The functor $j_{!*}$ is called the **Goresky-MacPherson extension** or **minimal (or intermediate) extension** functor.

Proposition 30.2. *The support of an irreducible D-module is irreducible.*

Proof. Let M be a D_X-module with support Z. Assume that Z is reducible: $Z = Z_1 \cup Z_2$ where Z_1 is an irreducible component of Z and Z_2 the union of all the other components. Let $Y = Z_1 \cap Z_2$, a proper subset in Z_1 and Z_2. Let $Z^\circ = Z \setminus Y$, $Z_i^\circ = Z_i \setminus Y$ and $X^\circ = X \setminus Y$. Then $Z^\circ = Z_1^\circ \cup Z_2^\circ$ is disconnected: Z_1°, Z_2° are closed nonempty subsets of Z° and $Z_1^\circ \cap Z_2^\circ = \emptyset$. Let M_1, M_2 be the sums of all subsheaves of $M|_{X^\circ}$ which are killed by localization away from Z_1°, respectively Z_2°. It is easy to show that M_i are nonzero submodules of $M|_{X^\circ}$ and $M|_{X^\circ} \cong M_1 \oplus M_2$. Thus $M|_{X^\circ}$ is reducible and hence so is M. □

30.3. **Kashiwara's theorem.** Let X be a smooth variety and $Z \subset X$ a smooth closed subvariety with closed embedding $i : Z \hookrightarrow X$. For $M \in \mathcal{M}(X)$ define M_Z to be the sheaf on X whose sections on an affine open set $U \subset X$ are the vectors in $M(U)$ annihilated by regular functions on U vanishing on Z. Thus the $\mathcal{O}(U)$-action on $M_Z(U)$ factors through $\mathcal{O}(Z \cap U)$. Also it is easy to see that $M_Z(U)$ depends only on $Z \cap U$, i.e., it gives rise to a quasicoherent sheaf $i^\dagger M$ on Z with sections

$$i^\dagger M(V) := M_Z(U)$$

for affine open $U \subset X$ such that $V = Z \cap U$. Moreover, if v is a vector field on U tangent to V then v preserves the ideal of V, hence acts naturally on $i^\dagger M(V)$. Furthermore, the action of v on this space depends only on the vector field on V induced by v. Thus $i^\dagger M(V)$ carries an action of the Lie algebra $\mathrm{Vect}(V)$. Together with the action of $\mathcal{O}(V)$, this defines an action of $D(V)$ on $i^\dagger M(V)$. We conclude that $i^\dagger M$ is naturally a D_Z-module. Thus we have defined a left exact functor

$$i^\dagger : \mathcal{M}(X) \to \mathcal{M}(Z).$$

It is called the **shifted inverse image** functor. This terminology is motivated by the following exercise.

Exercise 30.3. Show that $i^\dagger = L^d i^!$ and $i^! = R^d i^\dagger$, where L^d, R^d are the d-th left, respectively right derived functors and $d = \dim X - \dim Z$.

Theorem 30.4. *(Kashiwara) The functor $i^\dagger$ is an equivalence of categories $\mathcal{M}_Z(X) \to \mathcal{M}(Z)$.*

The proof is not difficult, but we will skip it (see [HTT]).

The inverse of the functor $i^\dagger$ is called the **direct image** functor and denoted $i_* : \mathcal{M}(Z) \to \mathcal{M}_Z(X)$, as it is a special case of the direct image functor defined above for affine morphisms. If we view i_* as a functor $\mathcal{M}(Z) \to \mathcal{M}(X)$ then it has both left and right adjoint, where are $i^!$ and $i^\dagger$, respectively.

Let us give a prototypical example.

Example 30.5. Let $X = \mathbb{A}^1$, $Z = \{0\}$. Then $\mathcal{M}(Z) = \mathrm{Vect}$ and $i_*(V) = V \otimes \delta_0$. So in this case Kashiwara's theorem reduces to the claim that $\mathrm{Ext}^1(\delta_0, \delta_0) = 0$.

Remark 30.6. We note that the above formalism and results extend in a straightforward manner to the case of twisted D-modules.

30.4. **Equivariant D-modules.** Let X be an algebraic variety with an action of an affine algebraic group G. Let us review the notion of a G-**equivariant quasicoherent sheaf** on X. Roughly speaking, this is a quasicoherent sheaf $\mathcal{E}$ on X equipped with a system of isomorphisms $\phi_g : g(\mathcal{E}) \cong \mathcal{E}, g \in G$ such that $\phi_{gh} = \phi_g \circ g(\phi_h)$ and ϕ_g depends on g algebraically. To give a formal definition, note that the group structure gives us a multiplication map $m : G \times G \twoheadrightarrow G$, and the action of G gives us a map $\rho : G \times X \twoheadrightarrow X$. We have a commutative diagram

$$\begin{tikzcd} & G \times G \times X \arrow[dl, "m\times \mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\rho"] & \\ G \times X \arrow[dr, "\rho"] & & G \times X \arrow[dl, "\rho"'] \\ & X & \end{tikzcd} .$$

Definition 30.7. A G-**equivariant quasicoherent sheaf** on X is a quasicoherent sheaf $\mathcal{E}$ on X equipped with an isomorphism

$$\phi : \rho^*\mathcal{E} \cong \mathcal{O}_G \boxtimes \mathcal{E}$$

making the following diagram commutative:

$$\begin{tikzcd} (\mathrm{id}\times\rho)^*\rho^*\mathcal{E} \arrow[r, "(\mathrm{id}\times\rho)^*\phi"] \arrow[d, equal] & (\mathrm{id}\times\rho)^*(\mathcal{O}_G \boxtimes \mathcal{E}) \arrow[r] & \mathcal{O}_G \boxtimes \rho^*\mathcal{E} \arrow[d, "\mathcal{O}_G\boxtimes\phi"] \\ (m\times\mathrm{id})^*\rho^*\mathcal{E} \arrow[r, "(m\times\mathrm{id})^*\phi"] & (m\times\mathrm{id})^*(\mathcal{O}_G \boxtimes \mathcal{E}) \arrow[r] & \mathcal{O}_G \boxtimes \mathcal{O}_G \boxtimes \mathcal{E} \end{tikzcd}$$

Thus ϕ comprises all the isomorphisms ϕ_g, which therefore satisfy the equality $\phi_{gh} = \phi_g \circ g(\phi_h)$ and depend on g algebraically.

We now wish to define the notion of a G-**equivariant D_X-module**. To this end, recall that for any D_X-module $\mathcal{E}$, the quasicoherent sheaf $\rho^*\mathcal{E}$ carries a natural structure of a $D_{G\times X}$-module (the D-module inverse image). We now make the following definition.

Definition 30.8. A **weakly G-equivariant D-module** on X is a D_X-module $\mathcal{E}$ with a G-equivariant quasicoherent sheaf structure, where ϕ is D_X-linear.

Note that if $\mathcal{E}$ is a weakly equivariant D_X-module then we have two (in general, different) actions of $\mathfrak{g} = \mathrm{Lie}(G)$ on $\mathcal{E}$. First of all, the G-action on X gives us maps $\mathfrak{g} \to \mathrm{Vect}(X) \to D(X)$, and so the D-module structure on $\mathcal{E}$ gives us a $\mathfrak{g}$-action $x \mapsto b_0(x)$ on $\mathcal{E}$. Note that this action does not depend on the choice of the weakly equivariant structure ϕ.

On the other hand, we have a $\mathfrak{g}$-action on $\mathcal{O}_G \boxtimes \mathcal{E}$ coming from the G-action on $G \times X$ given by $g \cdot (h, x) = (gh, x)$. Translating this along ϕ, we get a $\mathfrak{g}$-action on $\rho^*\mathcal{E}$. Restricting to $1 \times X$, this gives us another $\mathfrak{g}$-action $x \mapsto b_\phi(x)$ on $\mathcal{E}$.

Definition 30.9. A (strongly) G-**equivariant** D_X-**module** is a weakly G-equivariant D_X-module where these two $\mathfrak{g}$-actions agree: $b_\phi = b_0$ (or, equivalently, where ϕ is $D_{G\times X}$-linear.)

In general, since $[b_0(x), L] = [b_\phi(x), L]$ for $L \in D_X$, the operator $\rho_\phi(x) := b_\phi(x) - b_0(x)$ is a D-module endomorphism of $\mathcal{E}$. Moreover, it is easy to see that ρ_ϕ is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(\mathcal{E})$. In particular, if $\mathcal{E}$ is irreducible then by Dixmier's lemma (Lemma 7.2), $\mathrm{End}(\mathcal{E}) = \mathbb{C}$, so ρ_ϕ is just a character of $\mathfrak{g}$. Thus if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is perfect (for example, semisimple) then every weakly G-equivariant irreducible D_X-module is actually (strongly) G-equivariant.

Remark 30.10. A given D_X-module may have many weakly G-equivariant structures, but if G is connected, then it can only have one G-equivariant structure. This is because the $\mathfrak{g}$-action on $\mathcal{E}$ is determined by the map $\mathfrak{g} \to D(X)$ and this action can be integrated to a G-equivariant structure in an unique way (recall that we always work over a field of characteristic 0.)

Furthermore, any D_X-linear map of G-equivariant D_X-modules is automatically compatible with the G-action. This is because such a map is necessarily $\mathfrak{g}$-linear, which implies that it is in fact G-linear. These two facts combined show that the category of G-equivariant D_X-modules is a full subcategory of the category of D_X-modules. Stated another way, G-equivariance of a D_X-module is a property, not a structure.

Example 30.11. Consider the case where X is a point. Then $D_X \cong \mathbb{C}$ and so a D_X-module is a just a vector space. A weakly G-equivariant D_X-module is then simply a locally algebraic representation of G. This representation gives a G-equivariant structure if and only if $\mathfrak{g}$ acts by 0, i.e., the connected component of the identity $G_0 \subset G$ acts trivially. Thus a G-equivariant D_X-module is just a representation of the component group G/G_0. Conversely, any locally algebraic representation V of G gives rise to a weakly G-equivariant D-module on X which is equivariant iff G_0 acts trivially on V, so that V is a representation of G/G_0.

Example 30.12. Let $X = G/H$, where G is an algebraic group and H a closed subgroup of G. Then we claim that a G-equivariant D_X-module is the same thing as an H-equivariant D-module on a point, i.e., a representation of the component group H/H_0. Indeed, given an H/H_0-module V, we can define a G-equivariant vector bundle

$$(G \times V)/H \to X = G/H,$$

where H acts on $G \times V$ via $(g, v)h = (gh, h^{-1}v)$. Note that this can be written as $\frac{(G/H_0)\times V}{H/H_0}$ (as H_0 acts on V trivially). This shows that this vector bundle has a natural flat connection, i.e. is a D_X-module $L(X, V)$, which is clearly G-equivariant. The assignment $V \mapsto L(X, V)$ is the desired equivalence. In the case $H = G$, this reduces to Example 30.11.

Exercise 30.13. (i) Define the algebraic group $L := G \times_{G/G_0} H/H_0$ of pairs (g, h), $g \in G$, $h \in H/H_0$ which map to the same element of G/G_0; thus we have a short exact sequence

$$1 \to G_0 \to L \to H/H_0 \to 1.$$

Show that the category of weakly G-equivariant D-modules on G/H is naturally equivalent to the category of representations of L, such that the subcategory of strongly G-equivariant D-modules is identified with the subcategory of representations of L pulled back from the second factor H/H_0 (i.e., those with trivial action of G_0), and the subcategory of modules of the form $\mathcal{O}(G/H)\otimes V$ where V is a G-module is identified with the category of representations of L pulled back from the first factor G.

(ii) Let $\Delta : H \to L$ be the map defined by $\Delta(h) = (h, h)$. Show that the forgetful functor from weakly G-equivariant D-modules on G/H to G-equivariant quasicoherent sheaves on G/H corresponds to the pullback functor Δ^*.

Exercise 30.14. Let X be a smooth variety with an action of an affine algebraic group G and $H \subset G$ be a closed subgroup. Show that the category of H-equivariant D-modules on X is naturally equivalent to the category of G-equivariant D-modules on $X \times G/H$ with diagonal action of G (note that when X is a point, this reduces to Example 30.12).

Exercise 30.15. Let X be a principal G-bundle over a smooth variety Y. Show that the category of G-equivariant D_X-modules is naturally equivalent to the category of D_Y-modules. Namely, given a G-equivariant D_X-module M, for an affine open set $U \subset Y$ let $\widetilde{U}$ be the preimage of U in X and let $\overline{M}(U) := M(\widetilde{U})^G$. Then $\overline{M}$ is a D_Y-module, and the assignment $M \mapsto \overline{M}$ is a desired equivalence.

The notion of a weakly equivariant D-module often arises in the following setting. Let T be an algebraic torus and let $\widetilde{X}$ be a principal T-bundle over X.

Definition 30.16. A **monodromic D_X-module** (with respect to the bundle $\widetilde{X} \twoheadrightarrow X$) is a weakly T-equivariant $D_{\widetilde{X}}$-module.

Example 30.17. A monodromic D_X-module with $\rho_\phi = \lambda \in \mathrm{Lie}(T)^*$ is the same thing as a λ-twisted D-module on X, i.e., a $D_{\lambda,X}$-module.

Proposition 30.18. *Assume that X is a D-affine variety and that K is an affine algebraic group acting on X. Let $D(X)$ be the ring of global sections of D_X. Then the category of K-equivariant D_X-modules is equivalent to the category of $D(X)$-modules M endowed with a locally finite K-action whose differential coincides with the action of* $\mathrm{Lie}(K)$ *on M coming from the map* $\mathrm{Lie}(K) \to D(X)$.

Exercise 30.19. Prove Proposition 30.18.

In particular, by the Beilinson-Bernstein localization theorem, Proposition 30.18 applies to $X = \mathcal{F} \cong G/B$ and K a closed subgroup of G, and moreover it extends to the case of λ-twisted differential operators on $\mathcal{F}$ for antidominant $\lambda \in \mathfrak{h}^*$. Thus we get

Corollary 30.20. *If $\lambda \in \mathfrak{h}^*$ is antidominant then the functors* Γ, Loc *restrict to mutually inverse equivalences between the category of $(\mathfrak{g}, K)$-modules with infinitesimal character $\chi_{\lambda-\rho}$ and the category of K-equivariant D_λ-modules on $\mathcal{F}$.*

18.757 Representations of Lie Groups
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