31. **Applications of D-modules to representation theory**

31.1. **Classification of irreducible equivariant D-modules for actions with finitely many orbits.**

Theorem 31.1. *Let X be a smooth variety and K a connected algebraic group acting on X with finitely many orbits. Then there are finitely many irreducible K-equivariant D-modules on X. Namely, they are parametrized by pairs (O, V) where O is an orbit of K on X and V is an irreducible representation of the component group H/H_0 of the stabilizer $H := K_x$ for $x \in O$, $(O, V) \mapsto M(O, V)$.*

Proof. Let M be an irreducible K-equivariant D-module on X. Then by Proposition 30.2, the support Z of M is irreducible. Thus $Z = \overline{O}$ for a single orbit O of K. Let $Z_0 = \overline{O} \setminus O$, and $U = X \setminus Z_0$. Then U is a K-stable open subset of X and O is closed in U. Also $M|_U$ is a simple D_U-module supported on O. Let $i : O \hookrightarrow X$ be the closed embedding. By Kashiwara's theorem (Theorem 30.4) $i^\dagger M$ is a simple K-equivariant D-module on O. Thus by Example 30.12 $i^\dagger M = L(O, V)$ for some irreducible representation V of the component group of the stabilizer K_x, $x \in O$. Also it is clear that $L(O, V)$ gives rise to a simple K-equivariant D-module on X, namely, $M(O, V) := j_{!*} i_* M(O, V)$, where $j : U \hookrightarrow X$ is the open embedding. This proves the theorem. □

Remark 31.2. Theorem 31.1 can be extended in a straightforward way to weakly equivariant D-modules. In this case, recall that the weakly equivariant structure on an irreducible D-module M defines a character $\rho : \mathfrak{k} \to \mathbb{C}$, where $\mathfrak{k} = \mathrm{Lie}K$. Theorem 31.1 then holds with the only change: rather than being a representation of H/H_0, V now needs to be a representation of H in which $\mathrm{Lie}(H)$ acts by the character ρ. The proof is analogous to the case $\rho = 0$.

In particular, this applies to the case of twisted D-modules. In this case we have a principal T-bundle $p : \widetilde{X} \to X$ and a character $\lambda \in \mathfrak{t}^*$, $\mathfrak{t} = \mathrm{Lie}(T)$. Suppose K acts on X preserving this bundle; i.e., it acts on $\widetilde{X}$ and commutes with T. So we have a $K \times T$-action on $\widetilde{X}$ and a K-equivariant λ-twisted D-module on X is just a weakly $K \times T$-equivariant D-module on $\widetilde{X}$ with $\rho(\mathrm{k}, t) := \lambda(t)$. Now, for every K-orbit O on X, we have the stabilizer K_x, $x \in O$, and a homomorphism $\xi_x : K_x \to T$ defined by the condition that $(g, \xi_x(g))$ acts trivially on $p^{-1}(x)$ for $g \in K_x$. This defines a character $\lambda_x = \lambda \circ d\xi_x$ of $\mathrm{Lie}(K_x)$, and the simple K-equivariant D_λ-modules on X are $M(O, V)$ where V is an irreducible representation of K_x with $\mathrm{Lie}(K_x)$ acting by the character λ_x.

31.2. **Classification of irreducible Harish-Chandra modules.** Let $G_{\mathbb{R}}$ be a connected real semisimple algebraic group, $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ a maximal compact subgroup, $G, K \subset G$ their complexifications. By Corollary 30.20, if λ is antidominant then the Beilinson-Bernstein equivalence restricts to an equivalence between the category of $(\mathfrak{g}, K)$-modules with infinitesimal character $\chi_{\lambda-\rho}$ and the category of K-equivariant D_λ-modules on $\mathcal{F} = G/B$.

Proposition 31.3. *The group K acts on $\mathcal{F}$ with finitely many orbits.*

We will not give a proof of this proposition. For the proof and description of the set of orbits, see [RS].

Proposition 31.3 along with Theorem 31.1 allows us to classify irreducible $(\mathfrak{g}, K)$-modules (i.e., Harish-Chandra modules) for a regular infinitesimal character (the general case can be handled similarly).

Namely, let $H \subset B \subset G$ be a maximal torus and Borel subgroup of G; so $H \cong B/[B, B]$. Note that $K \times H$ acts on $\widetilde{\mathcal{F}} = G/[B, B]$. So for a K-orbit O on $\mathcal{F} = G/B$ and $x \in O$, we have a homomorphism $\xi_x : K_x \to H$ such that $(g, \xi_x(g))$ acts trivially on the fiber over x in $\widetilde{\mathcal{F}}$ for $g \in K_x$.

Let χ be a regular infinitesimal character for $\mathfrak{g}$ and λ be an antidominant weight with $\chi = \chi_{\lambda-\rho}$ (note that it always exists).

Theorem 31.4. *Irreducible $(\mathfrak{g}, K)$-modules with (pure) infinitesimal character χ are $\pi(O, V)$ where O is a K-orbit on $\mathcal{F}$ and V an irreducible representation of K_x, $x \in O$ such that $\mathrm{Lie}(K_x)$ acts via the character λ_x. Namely, $\pi(O, V)$ corresponds to $M(O, V)$ under the Beilinson-Bernstein equivalence.*

Example 31.5. Let $G_{\mathbb{R}} = SL_2(\mathbb{R})$. Let $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{Z}_{>0}$ and set $\chi = \chi_{\lambda-1}$ (so $\chi \neq \chi_0$). In this case $\mathcal{F} = \mathbb{CP}^1$ is the Riemann sphere, and $K = \mathbb{C}^\times$ acts by $\mathrm{k} \circ z := \mathrm{k}^2 z$. Thus we have three orbits: $0, \infty$, and $\mathbb{C}^\times$. For the orbit $\mathbb{C}^\times$ we have $K_x = \mathbb{Z}/2$, so we have two irreducible representations $V = \mathbb{C}_\pm$, which generically correspond to principal series representations $\pi(\mathbb{C}^\times, V_\pm) = P_\pm(1-\lambda)$ (see Section 9). The other two orbits have a connected stabilizer, and $\lambda_x = \pm\lambda$. Thus for such orbits representations exist only for $\lambda \in \mathbb{Z}_{\le 0}$. It is easy to see that these are exactly the discrete series representations $M^+_{\lambda-2}, M^-_{-\lambda+2}$. Also for such points one of the principal series representations is reducible ($P_+(1-\lambda)$ for even λ and $P_-(1-\lambda)$ for odd λ) and $\pi(\mathbb{C}^\times, V_+)$, respectively $\pi(\mathbb{C}^\times, V_-)$ is actually the finite-dimensional representation $L_{-\lambda}$. Thus we have four irreducible representations in this case.

Note that this agrees with our classification of irreducible representations of $SL_2(\mathbb{R})$ for regular infinitesimal characters discussed in Section 9.

Example 31.6. Let G be a simply connected complex semisimple group regarded as a real group. Then its maximal compact subgroup is G_c, so its complexification is G, and $G_{\mathbb{C}} = G \times G$, so that the inclusion $(G_c)_{\mathbb{C}} = G \hookrightarrow G_{\mathbb{C}} = G \times G$ is the diagonal embedding. The flag variety is $\mathcal{F} \times \mathcal{F} = G/B \times G/B$. Thus Harish-Chandra bimodules with infinitesimal character $(\chi_{\mu-\rho}, \chi_{\lambda-\rho})$ for antidominant λ, μ are $\pi(O, V)$ where O runs over orbits of G on $G/B \times G/B$ and V over appropriate representations of isotropy groups. Note that orbits of G on $G/B \times G/B$ are in a natural bijection with orbits of B on G/B, which are the Schubert cells C_w labeled by $w \in W$. One can check that the condition for existence of V on the orbit C_w is that $\lambda - w\mu$ is integral, and then V is unique (as the isotropy groups are connected in this case). Thus we find that the irreducible Harish-Chandra bimodules with such infinitesimal character are labeled by elements w such that $\lambda - w\mu \in P$, which agrees with the classification we obtained in Subsection 25.2.

Exercise 31.7. Classify irreducible Harish-Chandra modules for $SL_3(\mathbb{R})$ with a regular infinitesimal character.

Hint. Classify orbits of $SO_3(\mathbb{C})$ on $SL_3(\mathbb{C})/B$. This is equivalent to classification of flags in a 3-dimensional complex inner product space E under the action of $SO(E)$. Then classify possible representations V of the isotropy group for each orbit.

Remark 31.8. The K-orbits on G/B can be classified in explicit combinatorial terms. Together with Theorem 31.4, this leads to an alternative proof, using the localization theorem, of the **Langlands classification** of irreducible Harish-Chandra modules (obtained by Langlands in 1973 by a different method, 8 years before the localization theorem was proved, [La]). This classification requires a serious separate discussion which is beyond the scope of these notes.

31.3. **Applications to category $\mathcal{O}$.** Let us now see how this approach allows us to study category $\mathcal{O}$ for a semisimple Lie algebra $\mathfrak{g}$.

Consider the category $\mathcal{C}$ of weakly $B \times B$-equivariant finitely generated D-modules on G which are equivariant under $[B, B] \times [B, B]$ (it is easy to see that such modules have finite length). Thus for $M \in \mathcal{C}$, we have a homomorphism $\rho : \mathfrak{h} \oplus \mathfrak{h} \to \mathrm{End}(M)$, so $M = \oplus_{\mu,\lambda} M(\mu, \lambda)$ where $M(\mu, \lambda)$ is the generalized eigenspace for $\mathfrak{h} \oplus \mathfrak{h}$ with eigenvalue $(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*$. Thus we have a decomposition $\mathcal{C} = \oplus_{\mu,\lambda} \mathcal{C}_{\mu,\lambda}$.

Let $\mathcal{C}_{[\mu],\lambda}, \mathcal{C}_{\mu,[\lambda]}, \mathcal{C}_{[\mu],[\lambda]}$ be the full subcategories of $\mathcal{C}_{\mu,\lambda}$ consisting of objects on which the eigenvalues in square brackets are pure (without Jordan blocks). Thus we have

$$\mathcal{C}_{\mu,\lambda} \supset \mathcal{C}_{[\mu],\lambda}, \mathcal{C}_{\mu,[\lambda]} \supset \mathcal{C}_{[\mu],[\lambda]}$$

and all simple objects of $\mathcal{C}_{\mu,\lambda}$ are contained in $\mathcal{C}_{[\mu],[\lambda]}$. These objects are labeled by Bruhat cells $BwB \subset G$, $w \in W$ and representations V of the isotropy group satisfying an appropriate condition. As before, the condition for V to exist is that $\lambda - w\mu \in P$, thus $\mathcal{C}_{\mu,\lambda} = 0$ unless $\lambda - w\mu \in P$ for some $w \in W$.

We also see that $\mathcal{C}_{\lambda,\mu} \cong \mathcal{C}_{\lambda+\beta,\mu+\gamma}$ for $\beta, \gamma \in P$, and the same applies to its subcategories.

Let us now try to describe these categories representation-theoretically. To this end, note that we may interpret $\mathcal{C}_{[\mu],[\lambda]}$ as the category of weakly B-equivariant D_λ-modules on G/B with (pure) equivariance character μ. So if λ is antidominant, we get that $\mathcal{C}_{[\mu],[\lambda]}$ is equivalent to the full subcategory $\mathcal{O}_{[\mu],[\lambda]}$ of the category $\mathcal{O}_{\chi_{\lambda-\rho}}$ of objects with pure infinitesimal character $\chi_{\lambda-\rho}$ and weights in $\mu + P$. Similarly, $\mathcal{C}_{\mu,\lambda}, \mathcal{C}_{[\mu],\lambda}, \mathcal{C}_{\mu,[\lambda]}$ are equivalent to $\mathcal{O}_{\mu,\lambda}, \mathcal{O}_{[\mu],\lambda}, \mathcal{O}_{\mu,[\lambda]}$, where the corresponding (infinitesimal) character is pure if square brackets are present and generalized if not.

Now note that flipping left and right, we get equivalences $\mathcal{C}_{\lambda,\mu} \cong \mathcal{C}_{\mu,\lambda}$, $\mathcal{C}_{[\lambda],\mu} \cong \mathcal{C}_{\mu,[\lambda]}$ $\mathcal{C}_{\lambda,[\mu]} \cong \mathcal{C}_{[\mu],\lambda}$ $\mathcal{C}_{[\lambda],[\mu]} \cong \mathcal{C}_{[\mu],[\lambda]}$. If λ, μ are both antidominant, this yields equivalences of representation categories $\mathcal{O}_{\lambda,\mu} \cong \mathcal{O}_{\mu,\lambda}$, $\mathcal{O}_{[\lambda],\mu} \cong \mathcal{O}_{\mu,[\lambda]}$ $\mathcal{O}_{\lambda,[\mu]} \cong \mathcal{O}_{[\mu],\lambda}$ $\mathcal{O}_{[\lambda],[\mu]} \cong \mathcal{O}_{[\mu],[\lambda]}$. While the first equivalence is easy to see representation theoretically using translation functors, the others are not. They are clear from geometry but somewhat mysterious from the viewpoint of representation theory (although they can be understood using the Bernstein-Gelfand equivalence between category $\mathcal{O}$ and the category of Harish-Chandra bimodules, Theorem 25.8).

Example 31.9. If $\lambda, \mu \in P$, these categories are independent of λ, μ. Namely, let $\mathcal{O}_0$ be the category $\mathcal{O}$ for the trivial generalized infinitesimal character, and $\widetilde{\mathcal{O}}_0$ be its Serre closure (the category of modules admitting a finite filtration whose successive quotients are in $\mathcal{O}_0$; i.e. the action of $\mathfrak{h}$ is not necessarily diagonalizable but is only assumed locally finite). We may also define the category $\mathcal{O}_0^*$ of modules in $\widetilde{\mathcal{O}}_0$ which have pure infinitesimal character, and $\overline{\mathcal{O}}_0 \subset \mathcal{O}_0$ of modules with both pure infinitesimal character and diagonalizable action of $\mathfrak{h}$. Then the above four categories are exactly $\widetilde{\mathcal{O}}_0, \mathcal{O}_0, \mathcal{O}_0^*, \overline{\mathcal{O}}_0$. In particular, we obtain an equivalence $\mathcal{O}_0 \cong \mathcal{O}_0^*$ which is not obvious representation-theoretically.

Finally, we note that Exercise 30.14 applied to $X = G/B$ and $H = B$ gives a transparent geometric proof of Theorem 25.8.

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