18.781 Practice Questions for Final Exam

Note: The actual exam will be shorter (about 15 of these questions), in case you are timing yourself.

1. Find three solutions in positive integers of $|x^2 - 6y^2| = 1$ by first calculating the continued fraction expansion of $\sqrt{6}$.

2. If $\theta_1 = [3, 1, 5, 9, a_1, a_2, \ldots]$ and $\theta_2 = [3, 1, 5, 7, b_1, b_2, \ldots]$, show that $|\theta_1 - \theta_2| < 49/7095$.

3. For $n = 1728$, figure out the number of positive divisors of n, and the sum of its positive divisors.

4. Use multiplicativity to calculate the sum
$$\sum_{d|2592} \frac{\phi(d)}{d}.$$

5. Prove that if a prime p divides $n^2 - n + 1$ for an integer n, then $p \equiv 1 \pmod 6$.

6. Compute the value of the infinite periodic fraction $\langle 12, \overline{24} \rangle$. Find the smallest positive (i.e. both $x, y > 0$) solution of $x^2 - 145y^2 = 1$.

7. Determine whether there is a nontrivial integer solution of the equation
$$49x^2 + 5y^2 + 38z^2 - 28xy + 70xz - 28yz = 0.$$

8. Find a Pythagorean triangle such that the difference of the two (shorter) sides is 1, and every side is at least 100.

9. Show that $x^2 + 2y^2 = 8z + 5$ has no integral solution.

10. Define a sequence by $a_0 = 2$, $a_1 = 5$ and $a_n = 5a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Show that $a_n a_{n+2} - a_{n+1}^2$ is a square for every $n \geq 0$.

11. Let $p \nmid ab$. Show that $ax^2 + by^2 \equiv c \pmod p$ has a solution.

12. How many solutions are there to $x^2 + 3x + 18 \equiv 0 \pmod{28}$? Find all of them.

13. Let a, m be positive integers, not necessarily coprime. Show that $a^m \equiv a^{m-\phi(m)} \pmod m$.

14. Parametrize all the rational points on the curve $x^2 - 3y^2 = 1$.

15. Find an integer solution of $37x + 41y = -3$.

16. Show that if $n > 1$ then $n \nmid 2^n - 1$. (Hint: consider the smallest prime dividing n).

17. Let $p \geq 11$ be prime. Show that for some $n \in \{1, \ldots, 9\}$, both n and $n + 1$ are quadratic residues.

18. Show that if $23a^2 \equiv b^2 \pmod{17}$ then $23a^2 \equiv b^2 \pmod{289}$.

19. Calculate the product $\prod_\alpha (2 - \alpha)$, where α runs over the primitive 14'th roots of unity.

20. Let f be a multiplicative function with $f(1) = 1$, and let f^{-1} be its inverse for Dirichlet convolution. Show that f^{-1} is multiplicative as well, and that for squarefree n, we have $f^{-1}(n) = \mu(n)f(n)$.

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