1 Introduction

Most of the content of this overview lecture is contained in the slides that were used in class. These notes contain some additional details on using the Newton polygon to compute the genus of a plane curve. They imply, in particular, that all nonsingular cubics, including the Weierstrass equation $y^2 = x^3 + Ax + B$ with $-16(4A^3 + 27B^2) \neq 0$, are curves of genus 1, as are Edward's curves: $x^2 + y^2 = 1 + cx^2y^2$ with $c \neq 0, 1$.

1.1 Computing the genus of a plane curve

Let k be a field with algebraic closure $\bar{k}$. For a polynomial $f \in k[x, y]$ we use $f^* \in k[x, y, z]$ to denote its homogenization.

Definition 1.1. For a polynomial $f(x, y) = \sum a_{ij}x^iy^j \in k[x, y]$, the *Newton polygon* $\Delta(f)$ of f is the convex hull of the set $\{(i, j) : a_{ij} \neq 0\} \subseteq \mathbb{Z}^2$ in $\mathbb{R}^2$. The interior and boundary of $\Delta(f)$ are denoted $\Delta^\circ(f)$ and $\partial\Delta(f)$, respectively, and for each edge $\gamma \subseteq \Gamma\Delta(f)$ we define the polynomial $f_\gamma(x, y) := \sum_{(i,j)\in\gamma} a_{ij}x^iy^j$.

Theorem 1.2 (Baker's Theorem)**.** *Let $f(x, y) \in k[x, y]$ be irreducible in $\bar{k}[x, y]$, and let $F := \operatorname{Frac}(k[x, y]/(f))$ denote the corresponding function field, with genus $g(F)$. Then*

$$g(F) \leq \#\{\Delta^\circ(F) \cap \mathbb{Z}^2\}.$$

Proof. See [1, Theorem 2.4] for a short proof based on the Riemann–Roch theorem. □

Definition 1.3. A polynomial $f \in k[x, y]$ is *nondegenerate* with respect to an edge γ of $\partial\Delta(f)$ if the polynomials $f_\gamma, x\frac{\partial f_\gamma}{\partial x}, y\frac{\partial f_\gamma}{\partial y}$ have no common zero in $(\bar{k}^\times)^2$. The polynomial f is *nondegenerate* with respect to $\Delta(f)$ if it is nondegenerate with respect to every edge of $\partial\Delta(f)$ and not divisible by x or y.

Remark 1.4. For any edge γ of $\Delta(f)$, if either of the partial derivatives of $f_\gamma(x, y)$ is a monomial, then f is nondegenerate with respect to γ, since monomials have no zeros in $(\bar{k}^\times)^2$.

Proposition 1.5. *Let $f(x, y) \in k[x, y]$ be an irreducible nondegenerate polynomial in $\bar{k}[x, y]$, and suppose $f^*(x, y, z)$ has no singularities outside $\{(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)\}$. Then*

$$g(F) = \#\{\Delta^\circ(F) \cap \mathbb{Z}^2\}.$$

Proof. See [2, Theorem 4.2] □

Example 1.6. Let $f(x, y) = y^2 - x^3 - Ax + B$, with $A, B \in k$, and $-16(4A^3 + 27B^2) \neq 0$. Then $f(x, y)$ is irreducible in $\bar{k}[x, y]$, and $\partial\Delta(f)$ has the three edges $\gamma_1 = [(0, 0), (3, 0)]$, $\gamma_2 = [(0, 0), (0, 2)]$, and $\gamma_3 = [(0, 2), (0, 3)]$. We have

$$f_{\gamma_1}(x, y) = -x^3 - Ax - B,$$
$$f_{\gamma_2}(x, y) = y^2 - B,$$
$$f_{\gamma_3}(x, y) = y^2 - x^3.$$

The polynomial $f(x, y)$ is not divisible by x or y, and the fact that the discriminant of $x^3 + Ax + B$ is nonzero implies that f is nondegenerate with respect to γ_1. By Remark 1.4,

f is also nondegenerate with respect to the edges γ_2 and γ_3. Thus $f(x,y)$ is nondegenerate, and $f^*(x,y,z)$ has no singularities at all, so Proposition 1.5 implies that

$$g(F) = \#\{\Delta^0(F) \cap \mathbb{Z}^2\} = \#\{(1,1)\} = 1.$$

Example 1.7. Let $f(x,y) = x^2 + y^2 - 1 - cx^2y^2$ with $c \neq 0,1$. Then $f(x,y)$ is irreducible in $\bar{k}[x,y]$, and $\partial\Delta(f)$ has the four edges $\gamma_1 = [(0,0),(2,0)]$, $\gamma_2 = [(0,0),(0,2)]$, $\gamma_3 = [(0,2),(2,2)]$, and $\gamma_4 = [(2,0),(2,2)]$. We have

$$\begin{aligned} f_{\gamma_1}(x,y) &= x^2 - 1, \\ f_{\gamma_2}(x,y) &= y^2 - 1, \\ f_{\gamma_3}(x,y) &= y^2 - cx^2y^2, \\ f_{\gamma_4}(x,y) &= x^2 - cx^2y^2. \end{aligned}$$

The polynomial $f(x,y)$ is not divisible by x or y and Remark 1.4 applies to all four f_{γ_i}, thus f is nondegenerate. The homogenized polynomial $f^*(x,y,z)$ is singular only at $(0:1:0)$ and $(1:0:0)$, so f satsifies the hypothesis of Proposition 1.5 and

$$g(F) = \#\{\Delta^0(F) \cap \mathbb{Z}^2\} = \#\{(1,1)\} = 1.$$

References

[1] Peter Beelen, *A generalization of Baker's theorem*, Finite Fields and Their Applications **15** (2009), 558–568.

[2] Peter Beelen and Ruud Pellikaan, *The Newton polygon of plane curves with many rational points*, Designs, Codes and Cryptography **21** (2000), 41–67.