18.783 Elliptic Curves
Lecture 5

Andrew Sutherland

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Isogenies (Lecture 4 recap)

Definition

An isogeny $\alpha\colon E \to E'$ is a surjective morphism that is also a group homomorphism, equivalently, a non-constant rational map that sends zero to zero.

Lemma

If E and E' are elliptic curves over k in short Weierstrass form then every isogeny $\alpha\colon E \to E'$ can be put in standard form

$$\alpha(x,y) = \left(\frac{u(x)}{v(x)}, \frac{s(x)}{t(x)}y\right),$$

where $u,v,s,t \in k[x]$ are polynomials with $u \perp v$, $s \perp t$.
The roots of both v and t are the x-coordinates of the affine points in $\ker\alpha$.
The degree *of α is $\max(\deg u, \deg v)$, and α is* separable *if and only if $(u/v)' \neq 0$.*

Separable and inseparable isogenies

Lemma

Let k be a field of characteristic p. For relatively prime $u, v \in k[x]$ we have

$$(u/v)' = 0 \quad \Longleftrightarrow \quad u' = v' = 0 \quad \Longleftrightarrow \quad u = f(x^p) \text{ and } v = g(x^p) \text{ with } f, g \in k[x]$$

Proof

(first $\Leftrightarrow$): $(u/v)' = (u'v - v'u)/v^2 = 0$ iff $u'v = v'u$, and $u \perp v$ implies $u|u'$, which is impossible unless $u' = 0$, and similarly for v.
(second $\Leftrightarrow$): If $u = \sum_n a_n x^n$ then $u' = \sum n a_n x^n = 0$ iff $n a_n = 0$ for n with $a_n \neq 0$, in which case $u = \sum_m a_{mp} x^{mp} = f(x^p)$ where $f = \sum_m a_m x^m$, and similarly for v. $\square$

In characteristic zero the lemma says that $u' = 0$ if and only if $u \in k$, which means that every isogeny is separable, since isogenies are surjective morphisms.

Decomposing inseparable isogenies

Lemma

Let $\alpha\colon E \to E'$ *be an inseparable isogeny over* k *with* E *and* E' *in short Weierstrass form. Then* $\alpha(x, y) = \alpha(a(x^p), b(x^p)y^p)$ *for some* $a, b \in k(x)$*.*

Proof

This follows from the previous lemma, see Lemma 6.3 in the notes for details. □

Corollary

Isogenies of elliptic curves over a field of characteristic $p > 0$ *can be decomposed as*

$$\alpha = \alpha_{\text{sep}} \circ \pi^n,$$

for some separable α_{sep}*, with* $\pi\colon (x : y : z) \mapsto (x^p : y^p : z^p)$ *and* $n \geq 0$*.*
The separable degree *is* $\deg_s \alpha := \deg \alpha_{\text{sep}}$*, the* inseparable degree *is* $\deg_i \alpha := p^n$*.*

First isogeny-kernel theorem

Theorem

The order of the kernel of an isogeny is equal to its separable degree.

First isogeny-kernel theorem

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Corollary

A purely inseparable isogeny has trivial kernel.

Corollary

In any composition of isogenies $\alpha = \beta \circ \gamma$ *all degrees are multiplicative:*

$$\deg \alpha = (\deg \beta)(\deg \gamma), \qquad \deg_s \alpha = (\deg_s)(\deg_s \gamma), \qquad \deg_i \alpha = (\deg_i \beta)(\deg_i \gamma).$$

Second isogeny-kernel theorem

Definition

Let E/k be an elliptic curve. A subgroup G of $E(\bar{k})$ is defined over L/k if it is Galois stable, meaning $\sigma(G) = G$ for all $\sigma \in \mathrm{Gal}(\bar{k}/L)$.

Theorem

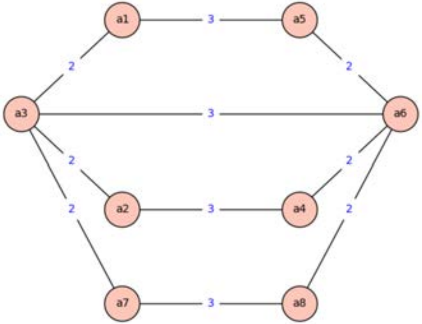
Let E/k be an elliptic curve and G a finite subgroup of $E(\bar{k})$ defined over k.
There is a separable isogeny $\alpha\colon E \to E'$ with kernel G.
The isogeny α and the elliptic curve E'/k are unique up to isomorphism.

Corollary

Isogenies of composite degree can be decomposed into isogenies of prime degree.

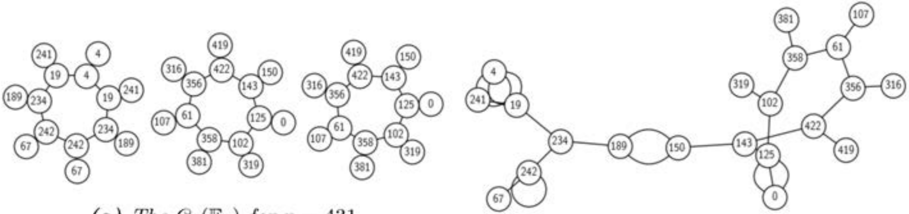
Isogeny graphs

$$\begin{pmatrix} 1 & 4 & 2 & 12 & 3 & 6 & 4 & 12 \\ 4 & 1 & 2 & 3 & 12 & 6 & 4 & 12 \\ 2 & 2 & 1 & 6 & 6 & 3 & 2 & 6 \\ 12 & 3 & 6 & 1 & 4 & 2 & 12 & 4 \\ 3 & 12 & 6 & 4 & 1 & 2 & 12 & 4 \\ 6 & 6 & 3 & 2 & 2 & 1 & 6 & 2 \\ 4 & 4 & 2 & 12 & 12 & 6 & 1 & 3 \\ 12 & 12 & 6 & 4 & 4 & 2 & 3 & 1 \end{pmatrix}$$



Isogeny class 30a in the L-functions and modular forms database.

Isogeny graphs



(a) The $\mathcal{G}_2(\mathbb{F}_p)$ for $p = 431$

(b) The spine $\mathcal{S} \subset \mathcal{G}_2(\overline{\mathbb{F}_p})$ for $p = 431$.

Figure 3.3: *Stacking, folding and attaching by an edge for* $\mathfrak{p} = 431$ *and* $\ell = 2$. *The leftmost component of* $\mathcal{G}_2(\mathbb{F}_p)$ *folds, the other two components stack, and the vertices* 189 *and* 150 *get attached by a double edge.*

Image taken from *Adventures in Supersingularland* by Sarah Arpin, Catalina Camacho-Navarro, Kristin Lauter, Joelle Lim, Kristina Nelson, Travis Scholl, and Jana Sotáková.

Isogeny graphs

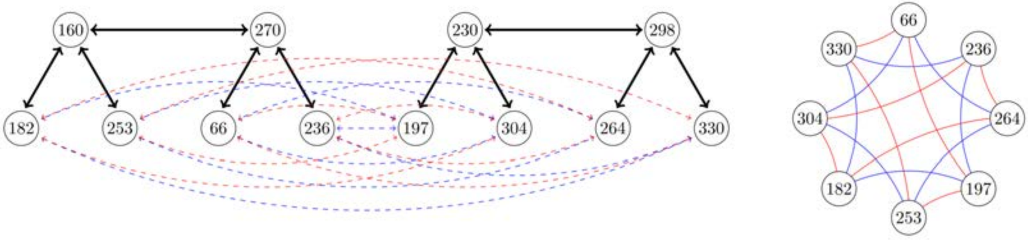


FIGURE 5. A whirlpool with two components.

Image taken from *Orienting supersingular isogeny graphs* by Leonardo Colò and David Kohel.

Instant poll

How many 2-isogenies does the elliptic curve $E\colon y^2 = f(x)$ admit?

A. Four, one for each point in $E[2] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

B. Three, one for each cyclic subgroup of order 2 in $E[2]$.

C. One for each rational point of order 2.

D. None if f is irreducible, one if f splits 1-2, three if f splits completely.

E. Infinitely many.

Constructing a separable isogeny from its kernel

Let E/k be an elliptic curve in Weierstrass form, and G a finite subgroup of $E(\bar{k})$. Let $G_{\neq 0}$ denote the set of nonzero points in G, which are affine points $Q = (x_Q, y_Q)$.

For affine points $P = (x_P, y_P)$ in $E(\bar{k})$ not in G define

$$\alpha(x_P, y_P) := \left(x_P + \sum_{Q \in G_{\neq 0}} (x_{P+Q} - x_Q)\,,\; y_P + \sum_{Q \in G_{\neq 0}} (y_{P+Q} - y_Q) \right).$$

Here x_P and y_P are variables, x_Q and y_Q are elements of $\bar{k}$, and x_{P+Q} and y_{P+Q} are rational functions of x_P and y_P giving coordinates of $P + Q$ in terms of x_P and y_P.

For $P \notin G$ we have $\alpha(P) = \alpha(P + Q)$ if and only if $Q \in G$, so $\ker \alpha = G$.

Vélu's formula for constructing 2-isogenies

Theorem (Vélu)

Let $E\colon y^2 = x^3 + Ax + B$ be an elliptic curve over k and let $x_0 \in \bar{k}$ be a root of $x^3 + Ax + B$. Define $t := 3x_0^2 + A$ and $w := x_0 t$. The rational map

$$\alpha(x,y) := \left(\frac{x^2 - x_0 x + t}{x - x_0},\ \frac{(x - x_0)^2 - t}{(x - x_0)^2} y \right)$$

is a separable isogeny from E to $E'\colon y^2 = x^3 + A'x + B'$, where $A' := A - 5t$ and $B' := B - 7w$. The kernel of α is the group of order 2 generated by $(x_0, 0)$.

If $x_0 \in k$ then E' and α will be defined over k, but in general E' and α will be defined over $k(A', B')$ which might be a quadratic or cubic extension of k.

Vélu's formula for constructing cyclic isogenies of odd degree

Theorem (Vélu)

Let $E\colon y^2 = x^3 + Ax + B$ be an elliptic curve over k and let G be a finite subgroup of $E(\bar{k})$ of odd order. For each nonzero $Q = (x_Q, y_Q)$ in G define

$$t_Q := 3x_Q^2 + A, \qquad u_Q := 2y_Q^2, \qquad w_Q := u_Q + t_Q x_Q,$$

$$t := \sum_{Q \in G_{\neq 0}} t_Q, \qquad w := \sum_{Q \in G_{\neq 0}} w_Q, \qquad r(x) := x + \sum_{Q \in G_{\neq 0}} \left. \frac{t_Q}{x - x_Q} + \frac{u_Q}{(x - x_Q)^2} \right).$$

The rational map

$$\alpha(x, y) := \bigl(r(x),\, r'(x)y\bigr)$$

is a separable isogeny from E to $E'\colon y^2 = x^3 + A'x + B'$, where $A' := A - 5t$ and $B' := B - 7w$, with $\ker \alpha = G$. If G is defined over k then so are α and E'.

Jacobian coordinates

Let us now work in the weighted projective plane, where x, y, z have weights $2, 3, 1$. This means, for example, that x^3 and y^2 are monomials of the same degree.

The homogeneous equation for an elliptic curve E in short Weierstrass form is then

$$y^2 = x^3 + axz^4 + Bz^6.$$

In general Weierstrass form we have

$$y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6,$$

Pro tip : a_i is the coefficient of the term containing z^i; this is why there is no a_5.

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In Jacobian coordinates the formulas for the group law look more complicated, but the formula for z_3 becomes very simple: $z_3 = x_1z_1^2 - x_2z_1^2$ when adding distinct points $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ and $z_3 = 2y_1z_1$ when doubling $(x_1 : y_1 : z_1)$.

Division polynomials

If we apply the group law in Jacobian coordinates to an affine point $P = (x : y : 1)$ on $E\colon y^2 = x^3 + Ax + B$ we can compute the rational map (in affine coordinates):

$$nP = \left(\frac{\phi_n}{{}^2_n}, \frac{\omega_n}{{}^3_n} \right).$$

where ϕ_n, ω_n, ψ_n are polynomials in $\mathbb{Z}[x, y, A, B]$ with degree at most 1 in y (we can reduce modulo $(y^2 - x^3 - Ax - B)$ to ensure this).

The polynomials ϕ_n and ψ_n^2 have degree 0 in y, so we write them as $\phi_n(x)$ and $\psi_n^2(x)$. Exactly one of ω_n and ψ_n^3 has degree 1 in y, so nP is effectively in standard form. (multiply the numerator by y^2 and the denominator by $x^3 + Ax + B$ if necessary).

Division polynomial recurrences

Definition

Let $E\colon y^2 = x^3 + Ax + B$ be an elliptic curve. Let $\psi_0 = 0$, and define $\psi_1, \psi_2, \psi_3, \psi_4$ as:

$$
\begin{aligned}
{}_1 &= 1, \\
{}_2 &= 2y, \\
{}_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2, \\
{}_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - A^3 - 8B^2).
\end{aligned}
$$

We then define ${}_n$ for $n > 4$ via the recurrences

$$
\begin{aligned}
{}_{2n+1} &= \psi_{n+2}\psi_n^3 - {}_{n-1}\psi_{n+1}^3, \\
{}_{2n} &= \frac{1}{2y}\psi_n(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2),
\end{aligned}
$$

We also define ${}_{-n} := -\psi_n$ (and the recurrences work for negative integers as well).

Division polynomial recurrences

Definition

Having defined $\;_n$ for $E\colon y^2 = x^3 + Ax + B$ and all $n \in \mathbb{Z}$, we now define

$$\begin{aligned}
\phi_n &:= x\psi_n^2 - \psi_{n+1}\psi_{n-1},\\
\omega_n &:= \frac{1}{4y}(\psi_{n+2}\psi_{n-1}^2 - \;_{n-2}\psi_{n+1}^2),
\end{aligned}$$

and one finds that $\phi_n = \phi_{-n}$ and $\omega_n = \omega_{-n}$.

It is a somewhat tedious algebraic exercise to verify that these recursive definitions agree with the definitions given by applying the group law. See this Sage notebook.

We rarely use ϕ_n and ω_n, but need to know the degree and leading coefficient of ϕ_n to compute the degree and separability of the multiplication-by-n map.

Multiplication-by-n maps

Theorem

Let E/k be an elliptic curve defined by the equation $y^2 = x^3 + Ax + B$ and let n be a nonzero integer. The multiplication-by-n map is defined by the affine rational map

$$[n](x, y) = \left(\frac{\phi_n(x)}{{}_n^2(x)}, \frac{\omega_n(x, y)}{{}_n^3(x, y)} \right)$$

Lemma

The polynomial $\phi_n(x)$ is monic of degree n^2 and the polynomial $\psi_n^2(x)$ has leading coefficient n^2, degree $n^2 - 1$, and is coprime to $\phi_n(x)$.

Corollary

The multiplication-by-n map on E/k has degree n^2 and is separable if and only $p \nmid n$.

Instant poll

Are you looking forward to class on Monday March 8?

A. Yes, I'm psyched to prove the structure theorem for torsion subgroups and learn about endomorphism rings!

B. No, I will be taking a well-earned break Monday, it's a student holiday.

C. No, but I am looking forward to class on Tuesday March 9, which is following a Monday schedule.

D. No, but I am looking forward to class on Wednesday March 10. Some time before then I plan to watch the recorded video of the lecture that would have taken place on March 8 if it were not a holiday.