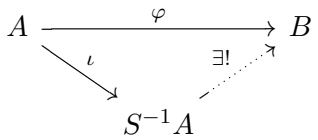
2 Localization and Dedekind domains

After a brief review of some commutative algebra background on localizations, in this lecture we begin our study of Dedekind domains, which are commutative rings that play a key role in algebraic number theory and arithmetic geometry (named after Richard Dedekind).

2.1 Localization of rings

Let A be a commutative ring (unital, as always), and let S be a multiplicative subset of A; this means S is closed under finite products (including the empty product, so $1 \in S$), and S does not contain zero. The *localization* of A with respect to S is a ring $S^{-1}A$ equipped with a ring homomorphism $\iota\colon A \to S^{-1}A$ that maps S into $(S^{-1}A)^\times$ and satisfies the following universal property: if $\varphi\colon A \to B$ is a ring homomorphism with $\varphi(S) \subseteq B^\times$ then there is a unique ring homomorphism $S^{-1}A \to B$ that makes the following diagram commute:



and one says that φ factors uniquely through $S^{-1}A$ (via ι). As usual with universal properties, this guarantees that $S^{-1}A$ is unique (hence well-defined), provided that it exists. To prove existence we construct $S^{-1}A$ as the quotient of $A \times S$ modulo the equivalence relation

$$(a, s) \sim (b, t) \Leftrightarrow \exists u \in S \text{ such that } (at - bs)u = 0. \tag{1}$$

We then use a/s to denote the equivalence class of (a, s) and define $\iota(a) := a/1$; one can easily verify that $S^{-1}A$ is a ring with additive identity $0/1$ and multiplicative identity $1/1$, and that $\iota : A \to S^{-1}A$ is a ring homomorphism. If s is invertible in A we can view a/s either as the element as^{-1} of A or the equivalence class of (a, s) in $S^{-1}A$; we have $(a, s) \sim (a/s, 1)$, since $(a \cdot 1 - a/s \cdot s) \cdot 1 = 0$, so this notation should not cause any confusion. For $s \in S$ we have $\iota(s)^{-1} = 1/s$, since $(s/1)(1/s) = s/s = 1/1 = 1$, thus $\iota(S) \subseteq (S^{-1}A)^\times$.

If $\varphi\colon A \to B$ is a ring homomorphism with $\varphi(S) \subseteq B^\times$, then $\varphi = \pi \circ \iota$, where π is defined by $\pi(a/s) := \varphi(a)\varphi(s)^{-1}$. If $\pi\colon S^{-1}A \to B$ is any ring homomorphism that satisfies $\varphi = \pi \circ \iota$, then $\varphi(a)\varphi(s)^{-1} = \pi(\iota(a))\pi(\iota(s))^{-1} = \pi(\iota(a)\iota(s)^{-1}) = \pi((a/1)(1/s)) = \pi(a/s)$, so π is unique.

In the case of interest to us, A is actually an integral domain, in which case $(a, s) \sim (b, t)$ if and only if $at - bs = 0$ (we can always take $u = 1$ in the equivalence relation (1) above), and we can then identify $S^{-1}A$ with a subring of the fraction field of A (which we note is the localization of A with respect to $S = A_{\neq 0}$), and if T is a multiplicative subset A that contains S, then $S^{-1}A \subseteq T^{-1}A$.

When A is an integral domain the map $\iota\colon A \to S^{-1}A$ is injective, allowing us to identify A with its image $\iota(A) \subseteq S^{-1}A$ (in general, ι is injective if and only if S contains no zero divisors). When A is an integral domain we may thus view $S^{-1}A$ as an intermediate ring that lies between A and its fraction field: $A \subseteq S^{-1}A \subseteq \operatorname{Frac} A$.

2.2 Ideals in localizations of rings

If $\varphi\colon A \to B$ is a ring homomorphism and $\mathfrak{b}$ is a B-ideal, then $\varphi^{-1}(\mathfrak{b})$ is an A-ideal called the *contraction* of $\mathfrak{b}$ to A and sometimes denoted $\mathfrak{b}^c$; when A is a subring of B and φ is

the inclusion map we simply have $\mathfrak{b}^c = \mathfrak{b} \cap A$. If $\mathfrak{a}$ is an A-ideal, in general $\varphi(\mathfrak{a})$ is not a B-ideal; but we can instead consider the B-ideal generated by $\varphi(\mathfrak{a})$, the *extension* of $\mathfrak{a}$ to B, sometimes denoted $\mathfrak{a}^e$.

In the case of interest to us, A is an integral domain, $B = S^{-1}A$ is the localization of A with respect to some multiplicative set S, and $\varphi = \iota$ is injective, so we view A as a subring of B. We then have

$$\mathfrak{a}^e = \mathfrak{a}B := (ab : a \in \mathfrak{a}, b \in B). \tag{2}$$

We clearly have $\mathfrak{a} \subseteq \varphi^{-1}((\varphi(\mathfrak{a}))) = \mathfrak{a}^{ec}$ and $\mathfrak{b}^{ce} = (\varphi(\varphi^{-1}(\mathfrak{b}))) \subseteq \mathfrak{b}$; one might ask whether these inclusions are equalities. In general the first is not: if $B = S^{-1}A$ and $\mathfrak{a} \cap S \neq \emptyset$ then $\mathfrak{a}^e = \mathfrak{a}B = B$ and $\mathfrak{a}^{ec} = B \cap A$ are both unit ideals, but we may still have $\mathfrak{a} \subsetneq A$. However when $B = S^{-1}A$ the second inclusion is an equality; see [1, Prop. 11.19] or [2, Prop. 3.11] for a short proof. We also note the following theorem.

Theorem 2.1. *Let S be a multiplicative subset of an integral domain A. There is a one-to-one correspondence between the prime ideals of $S^{-1}A$ and the primes ideals of A that do not intersect S given by the inverse maps $\mathfrak{q} \mapsto \mathfrak{q} \cap A$ and $\mathfrak{p} \mapsto \mathfrak{p}S^{-1}A$.*

Proof. See [1, Cor. 11.20] or [2, Prop. 3.11.iv]. □

Remark 2.2. An immediate consequence of (2) is that if $a_1, \dots, a_n \in A$ generate $\mathfrak{a}$ as an A-ideal, then they also generate $\mathfrak{a}^e = \mathfrak{a}B$ as a B-ideal. As noted above, when $B = S^{-1}A$ we have $\mathfrak{b} = \mathfrak{b}^{ce}$, so every B-ideal is of the form $\mathfrak{a}^e$ (take $\mathfrak{a} = \mathfrak{b}^c$). It follows that if A is noetherian then so are all its localizations, and if A is a PID then so are all of its localizations.

An important special case of localization occurs when $\mathfrak{p}$ is a prime ideal in an integral domain A, and $S = A - \mathfrak{p}$ (the complement of the set $\mathfrak{p}$ in the set A). In this case it is customary to denote $S^{-1}A$ by

$$A_{\mathfrak{p}} := \{a/b : a \in A, b \notin \mathfrak{p}\}/\sim, \tag{3}$$

and call it the *localization of A at $\mathfrak{p}$*. The prime ideals of $A_{\mathfrak{p}}$ are then in bijection with the prime ideals of A that lie in $\mathfrak{p}$. It follows that $\mathfrak{p}A_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ is therefore a local ring (whence the term *localization*).

Warning 2.3. The notation in (3) makes it tempting to assume that if a/b is an element of $\operatorname{Frac} A$, then $a/b \in A_{\mathfrak{p}}$ if and only if $b \notin \mathfrak{p}$. This is not necessarily true! As an element of $\operatorname{Frac} A$, the notation "a/b" represents an equivalence class; if $a/b = a'/b'$ with $b' \notin A_{\mathfrak{p}}$, then in fact $a/b = a'/b' \in A_{\mathfrak{p}}$. As a trivial example, take $A = \mathbb{Z}$, $\mathfrak{p} = (3)$, $a/b = 9/3$ and $a'/b' = 3/1$. You may object that we should write a/b in lowest terms, but when A is not a unique factorization domain it is not clear what this means.

Example 2.4. For a field k, let $A = k[x]$ and $\mathfrak{p} = (x - 2)$. Then

$$A_{\mathfrak{p}} = \{f \in k(x) : f \text{ is defined at } 2\}.$$

The ring A is a PID, so $A_{\mathfrak{p}}$ is a PID with a unique nonzero maximal ideal (the ideal $\mathfrak{p}A_{\mathfrak{p}}$), hence a DVR. Its maximal ideal is

$$\mathfrak{p}A_{\mathfrak{p}} = \{f \in k(x) : f(2) = 0\}.$$

The valuation on the field $k(x) = \operatorname{Frac} A$ corresponding to the valuation ring $A_{\mathfrak{p}}$ measures the order of vanishing of functions $f \in k(x)$ at 2. The residue field is $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq k$, and the quotient map $A_{\mathfrak{p}} \twoheadrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ sends f to $f(2)$.

Example 2.5. Let $p \in \mathbb{Z}$ be a prime. Then $\mathbb{Z}_{(p)} = \{a/b : a, b \in \mathbb{Z}, p \nmid b\}$. As in the previous example, $\mathbb{Z}$ is a PID and $\mathbb{Z}_{(p)}$ is a DVR; the valuation on $\mathbb{Q}$ is the p-adic valuation. The residue field is $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \simeq \mathbb{F}_p$ and the quotient map $\mathbb{Z}_{(p)} \twoheadrightarrow \mathbb{F}_p$ is reduction modulo p.

2.3 Localization of modules

The concept of localization generalizes immediately to modules. As above, let A be a commutative ring, let S a multiplicative subset of A, and let M be an A-module. The localization $S^{-1}M$ of M with respect to S is an $S^{-1}A$-module equipped with an A-module homomorphism $\iota\colon M \to S^{-1}M$ with the universal property that if N is an $S^{-1}A$-module and $\varphi\colon M \to N$ is an A-module homomorphism, then φ factors uniquely through $S^{-1}M$ (via ι). Note that in this definition we are viewing $S^{-1}A$-modules as A-modules via the canonical homomorphism $A \to S^{-1}A$ that is part of the definition of $S^{-1}A$. Our definition of $S^{-1}M$ reduces to the definition of $S^{-1}A$ in the case $M = A$.

The explicit construction of $S^{-1}M$ is exactly the same as $S^{-1}A$, one takes the quotient of the $S^{-1}A$-module $M \times S$ modulo the same equivalence relation as in (1):

$$(a, s) \sim (b, t) \Leftrightarrow \exists u \in S \text{ such that } (at - bs)u = 0,$$

where a and b now denote elements of M, and $\iota(a) := a/1$ as before. Alternatively, one can define $S^{-1}M := M \otimes_A S^{-1}A$ (see [2, Prop. 3.5] for a proof that this is equivalent). In other words, $S^{-1}M$ is the *base change* of M from A to $S^{-1}A$; we will discuss base change more generally in later lectures.

The map $\iota\colon M \to S^{-1}M$ is injective if and only if the map $M \xrightarrow{\times s} M$ is injective for every $s \in S$. This is a strong condition that does not hold in general, even when A is an integral domain (the annihilator of M may be non-trivial), but it applies to all the cases we care about. In particular, if A lies in a field K (in which case A must be an integral domain whose fraction field lies in K) and M is an A-module that is contained in a K-vector space. In this setting multiplication by any nonzero $s \in A$ is injective and we can view M as an A-submodule of any of its localizations $S^{-1}M$.

We will mostly be interested in the case $S = A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A, in which case we write $M_\mathfrak{p}$ for $S^{-1}M$, just as we write $A_\mathfrak{p}$ for $S^{-1}A$.

Proposition 2.6. *Let A be a subring of a field K, and let M be an A-module contained in a K-vector space V (equivalently, for which the map $M \to M \otimes_A K$ is injective).*[1] *Then*

$$M = \bigcap_\mathfrak{m} M_\mathfrak{m} = \bigcap_\mathfrak{p} M_\mathfrak{p},$$

where $\mathfrak{m}$ ranges over the maximal ideals of A, $\mathfrak{p}$ ranges over the prime ideals of A, and the intersections take place in V.

Proof. The fact that $M \subseteq \bigcap_\mathfrak{m} M_\mathfrak{m}$ is immediate. Now suppose $x \in \bigcap_\mathfrak{m} M_\mathfrak{m}$ and consider the A-ideal $\mathfrak{a} := \{a \in A : ax \in M\}$. For each maximal ideal $\mathfrak{m}$ we can write $x = m/s$ for some $m \in M$ and $s \in A - \mathfrak{m}$; we then have $sx \in M$ and $s \in \mathfrak{a}$, but $s \notin \mathfrak{m}$, so $\mathfrak{a} \not\subseteq \mathfrak{m}$. It follows that $\mathfrak{a}$ must be the unit ideal, so $1 \in \mathfrak{a}$ and $x = 1 \cdot x \in M$; thus $\cap_\mathfrak{m} M_\mathfrak{m} \subseteq M$.

We now note that each $M_\mathfrak{p}$ contains some $M_\mathfrak{m}$ (since each $\mathfrak{p}$ is contained in some $\mathfrak{m}$), and every maximal ideal is prime, so $\cap_\mathfrak{m} M_\mathfrak{m} = \cap_\mathfrak{p} M_\mathfrak{p}$. □

[1] The image is a tensor product of A-modules that is also a K-vector space. We need the natural map to be injective in order to embed M in it. Note that V necessarily contains a subspace isomorphic to $M \otimes_A K$.

An important special case of this proposition occurs when $K = \operatorname{Frac} A$ and $V = K$, in which case M is an A-submodule of K. Every ideal I of A is an A-submodule of K, and can thus be localized as above. The localization of I (as an A-module) at a prime ideal $\mathfrak{p}$ of A is the same thing as the extension of I (as an A-ideal) to the localization of A at $\mathfrak{p}$. In other words,

$$I_{\mathfrak{p}} = \{i/s : i \in I, s \in A - \mathfrak{p}\} = \{ia/s : i \in I, a \in A, s \in A - \mathfrak{p}\} = IA_{\mathfrak{p}}.$$

We also have the following corollary of Proposition 2.6.

Corollary 2.7. *Let A be an integral domain. Every ideal I of A (including $I = A$) is equal to the intersection of its localizations at the maximal ideals of A, and also to the intersection of its localizations at the prime ideals of A.*

Example 2.8. If $A = \mathbb{Z}$ then $\mathbb{Z} = \bigcap_p \mathbb{Z}_{(p)}$ in $\mathbb{Q}$.

Proposition 2.6 and Corollary 2.7 are powerful tools, because they allow us work in local rings (rings with just one maximal ideal), which often simplifies matters considerably. For example, to prove that an ideal I in an integral domain A satisfies a certain property, it is enough to show that this property holds for all its localizations $I_{\mathfrak{p}}$ at prime ideals $\mathfrak{p}$ and is preserved under intersections. We now want to consider rings A that satisfy some further assumptions that make its localizations become even easier to work with.

2.4 Dedekind domains

Proposition 2.9. *Let A be a noetherian domain. The following are equivalent:*

(i) *For every nonzero prime ideal $\mathfrak{p} \subset A$ the local ring $A_{\mathfrak{p}}$ is a DVR.*

(ii) *The ring A is integrally closed and* $\dim A \leq 1$.

Proof. If A is a field then (i) and (ii) both hold, so let us assume that A is not a field, and put $K := \operatorname{Frac} A$. We first show that (i) implies (ii). Recall that $\dim A$ is the supremum of the length of all chains of prime ideals. It follows from Theorem 2.1 that every chain of prime ideals $(0) \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ extends to a corresponding chain in $A_{\mathfrak{p}_n}$ of the same length; conversely, every chain in $A_{\mathfrak{p}}$ contracts to a chain in A of the same length. Thus

$$\dim A = \sup\{\dim A_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Spec} A\} = 1,$$

since every $A_{\mathfrak{p}}$ is either a DVR ($\mathfrak{p} \neq (0)$), in which case $\dim A_{\mathfrak{p}} = 1$, or a field ($\mathfrak{p} = (0)$), in which case $\dim A_{\mathfrak{p}} = 0$. Any $x \in K$ that is integral over A is integral over every $A_{\mathfrak{p}}$ (since they all contain A), and the $A_{\mathfrak{p}}$ are integrally closed, since they are DVRs or fields. So $x \in \bigcap_{\mathfrak{p}} A_{\mathfrak{p}} = A$, and therefore A is integrally closed, which shows (ii).

To show that (ii) implies (i), we first show that the following properties are all inherited by localizations of a ring: (1) no zero divisors, (2) noetherian, (3) dimension at most one, (4) integrally closed. (1) is obvious, (2) was noted in Remark 2.2, and (3) follows from Theorem 2.1 since, as argued above, we have $\dim A_{\mathfrak{p}} \leq \dim A$. To show (4), suppose $x \in K$ is integral over $A_{\mathfrak{p}}$. Then

$$x^n + \frac{a_{n-1}}{s_{n-1}}x^{n-1} + \cdots + \frac{a_1}{s_1}x + \frac{a_0}{s_0} = 0$$

for some $a_0, \dots, a_{n-1} \in A$ and $s_0, \dots, s_{n-1} \in A - \mathfrak{p}$. Multiplying both sides by s^n, where $s = s_0 \cdots s_{n-1} \in S$, shows that sx is integral over A, hence an element of A, since A is integrally closed. But then $sx/s = x$ is an element of $A_\mathfrak{p}$, so $A_\mathfrak{p}$ is integrally closed as claimed.

Thus (ii) implies that every $A_\mathfrak{p}$ is an integrally closed noetherian local domain of dimension at most 1, and for $\mathfrak{p} \neq (0)$ we must have $\dim A_\mathfrak{p} = 1$. Thus for every nonzero prime ideal $\mathfrak{p}$, the ring $A_\mathfrak{p}$ is an integrally closed noetherian local domain of dimension 1, and therefore a DVR, by Theorem 1.16. □

Definition 2.10. A noetherian domain satisfying either of the equivalent properties of Proposition 2.9 is called a *Dedekind domain*.

Corollary 2.11. *Every PID is a Dedekind domain. In particular, $\mathbb{Z}$ is a Dedekind domain, as is $k[x]$ for any field k.*

Remark 2.12. Every PID is both a UFD and a Dedekind domain. Not every UFD is a Dedekind domain (consider $k[x, y]$, for any field k), and not every Dedekind domain is a UFD (consider $\mathbb{Z}[\sqrt{-13}]$, in which $(1 + \sqrt{-13})(1 - \sqrt{-13}) = 2 \cdot 7 = 14$). However (as we shall see), every ring that is both a UFD and a Dedekind domain is a PID.

One of our first goals in this course is to prove that ring of integers of number fields and coordinate rings of global function fields are Dedekind domains. More precisely, we will prove that if A is a Dedekind domain and L is a finite separable extension of its fraction field, then the integral closure of A in L is a Dedekind domain. This includes the two main cases of interest to us, in which either $A = \mathbb{Z}$ and L is a number field, or $A = \mathbb{F}_q[t]$ and L is a global function field. Recall from Lecture 1 that number fields and global function fields are the two types of *global fields* (as we will prove in later lectures).

2.5 Fractional ideals

Throughout this subsection, A is a noetherian domain (not necessarily a Dedekind domain) and K is its fraction field.

Definition 2.13. A *fractional ideal* of a noetherian domain A is a finitely generated A-submodule of its fraction field.

Fractional ideals generalize the notion of an ideal: when A is noetherian the ideals of A are precisely the finitely generated A-submodules of A, and when A is also a domain we can extend this notion to its fraction field. Every ideal of A is also a fractional ideal of A, but fractional ideals are typically not ideals because they need not be contained in A. Some authors use the term *integral ideal* to distinguish the fractional ideals that lie in A (and are thus ideals) but we will not use this terminology.

Lemma 2.14. *Let A be a noetherian domain with fraction field K, and let $I \subseteq K$ be an A-module. Then I is finitely generated if and only if $aI \subseteq A$ for some nonzero $a \in A$.*

Proof. For the forward implication, if $r_1/s_1, \dots, r_n/s_n$ generate I as an A-module, then $aI \subseteq A$ for $a = s_1 \cdots s_n$. Conversely, if $aI \subseteq A$, then aI is an ideal, hence finitely generated (since A is noetherian), and if $a_1, \dots, a_n$ generate aI then $a_1/a, \dots, a_n/a$ generate I. □

Remark 2.15. Lemma 2.14 gives an alternative definition of fractional ideals that can be extended to domains that are not necessarily noetherian; they are A-submodules I of K for which there exists a nonzero $r \in A$ such that $rI \subseteq A$. When A is noetherian this coincides with our definition above.

Corollary 2.16. *Every fractional ideal of A can be written in the form $\frac{1}{a}I$, for some nonzero $a \in A$ and ideal I.*

Definition 2.17. A fractional ideal of A is *principal* if it is generated by one element, that is, it has the form xA for some $x \in K$. We will also use the notation $(x) \coloneqq xA$ to denote the principal fractional ideal generated by $x \in K$.

As with ideals, we can add and multiply fractional ideals:

$$I + J := (i + j : i \in I, j \in J), \qquad IJ := (ij : i \in I, j \in J).$$

Here the notation (S) means the A-module generated by $S \subseteq K$. As with ideals, we actually have $I + J = \{i + j : i \in I, j \in J\}$, but the ideal IJ is typically not the same as set $\{ij : i \in I, j \in J\}$, it consists of all finite sums of elements in this set. We also have a new operation, corresponding to division. For any fractional ideals I, J with J nonzero, the set

$$(I : J) := \{x \in K : xJ \subseteq I\}$$

is called a *colon ideal*. Some texts refer to $(I : J)$ as the *ideal quotient* of I by J, but note that it is **not** a quotient of A-modules (for example, $(\mathbb{Z} : \mathbb{Z}) = \mathbb{Z}$ but $\mathbb{Z}/\mathbb{Z} = \{0\}$).

We do not assume $I \subseteq J$ (or $J \subseteq I$), the definition makes sense for any fractional ideals I and J with J nonzero.[2] If $I = (x)$ and $J = (y)$ are principal fractional ideals then $(I : J) = (x/y)$, so colon ideals can be viewed as a generalization of division in $K^\times$.

Lemma 2.18. *Let I and J be fractional ideals of a noetherian domain A with J nonzero. Then $(I : J)$ is a fractional ideal of A.*

Proof. It is clear from the definition that $(I : J)$ is closed under addition and multiplication by elements of A (since I is), so $(I : J)$ is an A-module of the fraction field of A. To show that $(I : J)$ is finitely generated, we first suppose that $I, J \subseteq A$ are ideals. For any nonzero $j \in J \subseteq A$ we have $j(I : J) \subseteq I \subseteq A$, so $(I : J)$ is finitely generated, by Lemma 2.14. For the general case, choose a and b so that $aI \subseteq A$ and $bJ \subseteq A$ via Lemma 2.14. Then $(I : J) = (abI : abJ)$ with $abI, abJ \subseteq A$, which we have already shown is finitely generated. □

Definition 2.19. A fractional ideal I is *invertible* if $IJ = A$ for some fractional ideal J.

Inverses are unique when they exist: if $IJ = A = IJ'$ then $J = JA = JIJ' = AJ' = J'$. We may use I^{-1} to denote the inverse of a fractional ideal I when it exists.

Lemma 2.20. *A fractional ideal I of A is invertible if and only if $I(A : I) = A$ (in which case $(A : I)$ is its inverse).*

Before proving the lemma, note that $I(A : I) \subseteq A$ always holds, since for $y \in I$ and $x \in (A : I)$ we have $xy \in xI \subseteq A$, by the definition of $(A : I)$. The lemma states that this inclusion is an equality precisely when I is invertible.

[2] The definition still makes sense when J is the zero ideal, but $(I : (0)) = K$ will typically not be finitely generated as an A-module, hence not a fractional ideal.

Proof. Suppose I is invertible, with $IJ = A$. Then $jI \subseteq A$ for all $j \in J$, so $J \subseteq (A : I)$, and $A = IJ \subseteq I(A : I) \subseteq A$, so $I(A : I) = A$. □

In the next lecture we will prove that in a Dedekind domain every nonzero fractional ideal is invertible, but let us first note that this is not true in general.

Example 2.21. Consider the subring $A := \mathbb{Z}+2i\mathbb{Z}$ of the Gaussian integers (with $i^2 = -1$). The set $I := 2\mathbb{Z}[i]$ is a non-invertible A-ideal (even though it is an invertible $\mathbb{Z}[i]$-ideal); indeed, we have $(A : I) = \mathbb{Z}[i]$ and $I(A : I) = 2\mathbb{Z}[i] \subsetneq A$.

2.6 Invertible fractional ideals and the ideal class group

In this section A is a noetherian domain (not necessarily a Dedekind domain) and K is its fraction field. Recall that a fractional ideal of A is a finitely generated A-submodule of K, and if I and J are fractional ideals, so is the colon ideal

$$(I : J) := \{x \in K : xJ \subseteq I\},$$

and we say that a fractional ideal I is invertible if $IJ = A$ for some fractional ideal J. The definition of $(A : I)$ implies $I(A : I) \subseteq A$, and Lemma 2.20 implies that I is invertible precisely when this inclusion is an equality, in which case the inverse of I is $(A : I)$.

Ideal multiplication is commutative and associative, thus the set of nonzero fractional ideals of a noetherian domain form an abelian monoid under multiplication with $A = (1)$ as the identity. It follows that the subset of invertible fractional ideals is an abelian group.

Definition 2.22. The *ideal group* $\mathcal{I}_A$ of a noetherian domain A is the group of invertible fractional ideals. Note that, despite the name, elements of $\mathcal{I}_A$ need not be ideals.

Every nonzero principal fractional ideal (x) is invertible (since $(x)^{-1} = (x^{-1})$), and a product of principal fractional ideals is principal (since $(x)(y) = (xy)$), as is the unit ideal (1), thus the set of nonzero principal fractional ideals $\mathcal{P}_A$ is a subgroup of $\mathcal{I}_A$.

Definition 2.23. Let A be a noetherian domain. The quotient $\mathrm{cl}(A) := \mathcal{I}_A/\mathcal{P}_A$ is the *ideal class group* of A; it is also called the *Picard group* of A and denoted $\mathrm{Pic}(A)$.[3]

Example 2.24. If A is a DVR with uniformizer π then its nonzero fractional ideals are the principal fractional ideals (π^n) with $n \in \mathbb{Z}$ (including $n \le 0$). We have $(\pi^m)(\pi^n) = (\pi^{m+n})$, thus the ideal group of A is isomorphic to $\mathbb{Z}$ (under addition). In this case $\mathcal{P}_A = \mathcal{I}_A$ and the ideal class group $\mathrm{cl}(A)$ is trivial.

Remark 2.25. A Dedekind domain is a UFD if and only if its ideal class group is trivial (we will prove this in the next lecture), thus $\mathrm{cl}(A)$ may be viewed as a measure of how far A is from being a UFD. More generally, the ideal class group of an integrally closed noetherian domain A is trivial when A is a UFD, and the converse holds if one replaces the ideal class group with the *divisor class group*. One defines a divisor as an equivalence class of fractional ideals modulo the equivalence relation $I \sim J \Leftrightarrow (A : I) = (A : J)$, and in an integrally closed noetherian domain A (or more generally, a Krull domain), the set

[3] In general, the Picard group of a commutative ring A as the group of isomorphism classes of A-modules that are invertible under tensor product (equivalently, projective modules of rank one). When A is a noetherian domain, the Picard group of A is canonically isomorphic to the ideal class group of A and the two notions may be used interchangeably.

of divisors forms a group that contains principal divisors as a subgroup; the divisor class group is defined as the quotient, and it is trivial if and only if A is a UFD (this holds more generally for any Krull domain, see [5, Thm. 8.34]). In a Dedekind domain, fractional ideals are always distinct as divisors and every nonzero fractional ideal is invertible, so the ideal class group and divisor class group coincide.[4]

References

[1] Allen Altman and Steven Kleiman, *A term of commutative algebra*, Worldwide Center of Mathematics, 2013.

[2] Michael Atiyah and Ian MacDonald, *Introduction to commutative algebra*, Addison–Wesley, 1969.

[3] Pete L. Clark, *Commutative algebra*, 2015.

[4] Anthony W. Knapp, *Advanced Algebra*, Digital Second Edition, 2016.

[5] Max D. Larsen and Paul J. McCarthy, *Multiplicative thoery of ideals*, Academic Press, 1971.

[4] In general, the divisor class group and the ideal class group (or Picard group) of an integrally closed noetherian domain A may differ when $\dim A > 1$; see [3, Thm. 19.38] for a dimension 2 an example in which the ideal class group is trivial but the divisor class group is not (implying that A is not a UFD).

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