LECTURE 13

Homotopy Coinvariants, Abelianization, and Tate Cohomology

Recall that last time we explicitly constructed the homotopy invariants $X^{\mathrm{h}G}$ of a complex X of G-modules. To do this, we constructed the *bar resolution* $P_G^{\mathrm{can}} \xrightarrow{\mathrm{qis}} \mathbb{Z}$, where P_G^{can} is a canonical complex of free G-modules in non-positive degrees. Then we have a quasi-isomorphism $X^{\mathrm{h}G} \simeq \underline{\mathrm{Hom}}_G(P_G^{\mathrm{can}}, X)$.

In particular, we have

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathbb{Z}[G^3] & \longrightarrow & \mathbb{Z}[G\times G] & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

for P_G^{can}, with differential of the form $(g_1, g_2) \mapsto g_1 g_2 - g_1$ (for d^{-1}; the G-action is always on the first term). Note that if G is finite, then these are all finite-rank G-modules.

For every G-module M, we have

$$\cdots \to 0 \to M \xrightarrow{m \mapsto (gm-m)_{g\in G}} \underbrace{\prod_{g\in G} M}_{\{\varphi\colon G\to M\}} \to \prod_{g,h\in G} M \to \cdots$$

via some further differential, for $M^{\mathrm{h}G}$. We can use this expression to explicitly compute the first cohomology of $M^{\mathrm{h}G}$. It turns out that a function $\varphi\colon G \to M$ is killed by this differential if it is a 1-*cocycle* (sometimes called a *twisted homomorphism*), that is, $\varphi(gh) = \varphi(g) + g\cdot\varphi(h)$ for all $g, h \in G$ via the group action. Similarly, φ is a 1-*coboundary* if there exists some $m \in M$ such that $\varphi(g) = g\cdot m - m$ for all $g \in G$. The upshot is that

$$H^1(G, M) := H^1(M^{\mathrm{h}G}) = \{\text{1-cocycles}\}/\{\text{1-coboundaries}\}.$$

As a corollary, if G acts trivially on M, then $H^1(G, M) = \mathrm{Hom}_{\mathsf{Group}}(G, M)$, since the 1-coboundaries are all trivial, and the 1-cocycles are just ordinary group homomorphisms. This also shows that zeroth cohomology is just the invariants, as we showed last lecture.

Now, our objective (from a long time ago) is to define Tate cohomology and the Tate complex for any finite group G. We'd like $\hat{H}^0(G, M) = M^G/\mathrm{N}(M) = \mathrm{Coker}(M_G \xrightarrow{N} M^G)$, because it generalizes the central problem of local class field theory for extensions of local fields. Recall that $M_G = M/(g-1)$ (equivalent to tensoring with the trivial module, and dual to invariants, which we prefer as a submodule), so that this map factors through M and induced the norm map above.

Our plan is, for a complex X of G-modules, to form

$$X_{\mathrm{h}G} \xrightarrow{\mathrm{N}} X^{\mathrm{h}G} \to X^{\mathrm{t}G} := \mathrm{hCoker}(\mathrm{N}).$$

Thus, we first need to define the homotopy coinvariants $X_{\mathrm{h}G}$.

Note that if M is a G-module, then $M_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Define $I_G := \mathrm{Ker}(\epsilon)$, so that we have a short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$
$$\sum_i n_i g_i \mapsto \sum_i n_i,$$

We claim that I_G is $\mathbb{Z}$-spanned by $\{g-1 : g \in G\}$ (which we leave as an exercise). A corollary is that

$$\mathbb{Z}[G]^{\oplus G} \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

is exact, since $\mathbb{Z}[G]^{\oplus G} \twoheadrightarrow I_G$ via $1 \mapsto g-1$ on the gth coordinate.

Remark 13.1. The correct algorithm for computing tensor products is as follows: recall that tensor products are right-exact, that is, they preserve surjections, and tensoring with the algebra gives the original module. To tensor with a module, take generators and relations for that module, use it to write a resolution as above, tensor with that resolution, giving a matrix over a direct sum of copies of that module, and then take the cokernel.

It would be very convenient if we could define $M_{\mathrm{h}G}$ via an analogous process for chain complexes.

Definition 13.2. If X and Y are chain complexes, then

$$(X \otimes Y)^i := \bigoplus_{j \in \mathbb{Z}} X^j \otimes Y^{i-j},$$

with differential

$$d(x \otimes y) := dx \otimes y + (-1)^j x \otimes dy$$

If X is a complex of right A-modules, and Y is a complex of left A-modules, then $X \otimes_A Y$ is defined similarly.

Note that the factor of $(-1)^j$ ensures that the differential squares to zero. Also, there is no need to worry about left and right A-modules for algebras, since left and right algebras are isomorphic via changing the order of multiplication; for G-modules, this means replacing every element with its inverse.

Now, a bad guess for $X_{\mathrm{h}G}$ would be $X \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, because it doesn't preserve acyclic complexes, equivalently quasi-isomorphisms. A better guess is to take a projective resolution $P_G \simeq \mathbb{Z}$, e.g. P_G^{can}, and tensor with that instead: $X_{\mathrm{h}G} := X \otimes_{\mathbb{Z}[G]} P_G$.

Definition 13.3. A complex F of left A-modules is *flat* is for every acyclic complex Y of right A-modules, $Y \otimes_A F$ is also acyclic, that is, $- \otimes_A F$ preserves injections.

We now ask if P_G is flat. In fact:

Claim 13.4. *Any projective complex is flat.*

An easier claim is the following:

Claim 13.5. *Any complex F that is bounded above with F^i flat for all i is flat.*

To prove this claim, we will use the fact that projective modules are flat, as they are direct summands of free modules, which are trivially flat (i.e., if $F = F_1 \oplus F_2$, then $F \otimes M = (F_1 \otimes M) \oplus (F_2 \otimes M)$).

Proof. **Case 1.** Suppose F is in degree 0 only, i.e., $F^i = 0$ for all $i \neq 0$. For every complex $Y = Y^\bullet$, we have

$$\cdots \to Y^i \otimes_A F \xrightarrow{d^i \otimes \mathrm{id}_F} Y^{i+1} \otimes_A F \to \cdots$$

for $Y \otimes_A F$. Since F is flat, we have $H^i(Y \otimes_A F) = H^i(Y) \otimes_A F$ for each i (since F flat means that tensoring with F commutes with forming kernels, cokernels and images), so if Y is acyclic, then $Y \otimes_A F$ is as well.

Case 2. Suppose F is in degrees 0 and -1 only, i.e., F is of the form

$$\cdots \to 0 \to F^{-1} \to F^0 \to 0 \to \cdots,$$

and so $F^\bullet = \mathrm{hCoker}(F^{-1} \to F^0)$. Then since tensor products commute with homotopy cokernels, we obtain

$$Y \otimes_A F = \mathrm{hCoker}(Y \otimes_A F^{-1} \to Y \otimes_A F^0),$$

so by Case 1, if Y is acyclic, then $Y \otimes_A F^0$ and $Y \otimes_A F^{-1}$ are as well, hence $Y \otimes_A F$ is as well by the long exact sequence on cohomology. A similar (inductive) argument gives the case where F is bounded.

Case 3. In the general case, form the diagram

$$\begin{array}{ccccccccccccc}
F_0 & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F^0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \\
F_1 & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F^1 & \xrightarrow{d} & F^0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \\
F_2 & & \cdots & \longrightarrow & 0 & \longrightarrow & F^2 & \xrightarrow{d} & F^1 & \xrightarrow{d} & F^0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

Clearly all squares of this diagram commute, hence these are all morphisms of complexes, and $F = \varinjlim_i F_i$. Since direct limits commute with tensor products (note that is not true for inverse limits because of surjectivity), we have $Y \otimes_A F = \varinjlim_i Y \otimes_A F_i$. By Case 2, $Y \otimes_A F_i$ is acyclic for each i, so since cohomology commutes with direct limits (because they preserve kernels, cokernels, and images), if Y is acyclic, then $Y \otimes_A F$ is too. □

Remark 13.6. Let Y be a complex of A-modules, choose a quasi-isomorphism $F \xrightarrow{\mathrm{qis}} Y$, where F is flat, and define $Y \otimes_A^{\mathrm{der}} X := F \otimes_A X$. Then this is well-defined up to quasi-isomorphism, which is well-defined up to homotopy, etc. (it's turtles all the way down!).

Definition 13.7. The *ith torsion group (of Y against X)* is $\mathrm{Tor}_i^A(Y, X) := H^{-i}(Y \otimes_A^{\mathrm{der}} X)$.

Definition 13.8. The *homotopy coinvariants* of a chain complex X is the complex $X_{\mathrm{h}G} := X \otimes_{\mathbb{Z}[G]}^{\mathrm{der}} \mathbb{Z} \simeq X \otimes_{\mathbb{Z}[G]} P_G$ (which we note is only well-defined up to quasi-isomorphism).

DEFINITION 13.9. $H_i(G,X) := H^{-i}(X_{hG})$ (where we note that the subscript notation is preferred as X_{hG} is generally a complex in non-positive degrees only).

We now perform some basic calculations.

CLAIM 13.10. *If X is bounded from above by 0, then $H_0(G,X) = H^0(X)_G$ (the proof is similar to that of Claim 12.5).*

CLAIM 13.11. *$H_1(G,\mathbb{Z}) = G^{ab}$, where G^{ab} denotes the abelianization of G.*

Note that this is sort of a dual statement to what we saw at the beginning of lecture; $H^1(G,M)$ had to do with maps $G \to M$, which are the same as maps from $G^{ab} \to M$, and here $H_1(G,\mathbb{Z})$ is determined by the maps out of G.

PROOF. Recall the short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

The long exact sequence on cohomology gives an exact sequence

$$H_1(G,\mathbb{Z}[G]) \to H_1(G,\mathbb{Z}) \to H_0(G,I_G) \to H_0(G,\mathbb{Z}[G]) \to H_0(G,\mathbb{Z}).$$

We have

$$H_0(G,\mathbb{Z}[G]) = H^0(\mathbb{Z}[G])_G = \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}$$

by Claim 13.10. Certainly $H_0(G,\mathbb{Z}) = H^0(\mathbb{Z})_G = \mathbb{Z}$, and $H_1(G,\mathbb{Z}[G]) = 0$ as

$$\mathbb{Z}[G]_{hG} := \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} P_G = P_G \simeq \mathbb{Z}$$

is a quasi-isomorphism. Thus, our exact sequence is really

$$0 \to H_1(G,\mathbb{Z}) \xrightarrow{\sim} H_0(G,I_G) \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z},$$

which gives the noted isomorphism. The upshot is that

$$H_1(G,\mathbb{Z}) = (I_G)_G = I_G/I_G^2$$

since $M_G = M/I_G \cdot M$.

CLAIM 13.12. *The map*

$$\mathbb{Z}[G]/I_G^2 \to G^{ab} \times \mathbb{Z}, \quad g \mapsto (\bar{g}, 1)$$

is an isomorphism.

This would imply that $I_G/I_G^2 = \mathrm{Ker}(\epsilon)/I_G^2 = G^{ab}$, as desired.

PROOF. First note that the map above is a homomorphism. Indeed, letting $[g] \in \mathbb{Z}[G]$ denote the class of g, we have

$$[g] + [h] \mapsto (\bar{g}\bar{h}, 2)$$
$$[g] \mapsto (\bar{g}, 1)$$
$$[h] \mapsto (\bar{h}, 1)$$

for any $g, h \in G$, and the latter two images add up to the first. We claim that this map has an inverse, induced by the map

$$G \times \mathbb{Z} \to \mathbb{Z}[G]/I_G^2, \quad (g,n) \mapsto [g] + n - 1.$$

This is a homomorphism, as

$$([g]-1)([h]-1) = [gh] - [g] - [h] + 1 \in I_G^2,$$

and therefore
$$([g]-1)+([h]-1) \equiv [gh]-1 \bmod I_G^2,$$
as desired. Finally, they are inverses, as
$$(\bar{g},1) \mapsto [g]+1-1=[g] \quad \text{and} \quad [g]+n-1 \mapsto (\bar{g},1)(1,n-1)=(\bar{g},n),$$
as desired. □

This proves the claim. □

Finally, we define the norm map $X_{\mathrm{h}G} \xrightarrow{\mathrm{N}} X^{\mathrm{h}G}$ to be the composition
$$X_{\mathrm{h}G} = X \otimes_{\mathbb{Z}[G]} P_G \to X \otimes_{\mathbb{Z}[G]} \mathbb{Z} \to \underline{\mathrm{Hom}}_{\mathbb{Z}[G]}(\mathbb{Z}, X) \to \underline{\mathrm{Hom}}_{\mathbb{Z}[G]}(P_G, X) = X^{\mathrm{h}G},$$
where the second map is via degree-wise norm maps (using tensor-hom adjunction). We then set
$$X^{\mathrm{t}G} := \mathrm{hCoker}(X_{\mathrm{h}G} \xrightarrow{\mathrm{N}} X^{\mathrm{h}G}),$$
which we claim generalizes what we had previously for cyclic groups up to quasi-isomorphism, so that we may define
$$\hat{H}^i(G,X) := H^i(X^{\mathrm{t}G}).$$
Soon we will prove:

Claim 13.13 (lcft). *For a finite group G and extension L/K of local fields,*
$$P_G \to L^\times[2]$$
is an isomorphism on Tate cohomology.

This gives that
$$\hat{H}^{-2}(G,\mathbb{Z}) \simeq \hat{H}^0(G,L^\times) = K^\times/\mathrm{N}(L^\times).$$
We have an exact sequence
$$0 = H^{-2}(\mathbb{Z}^{\mathrm{h}G}) \to \hat{H}^{-2}(G,\mathbb{Z}) \xrightarrow{\sim} \underbrace{H^{-1}(\mathbb{Z}_{\mathrm{h}G})}_{H_1(G,\mathbb{Z})=G^{\mathrm{ab}}} \to H^{-1}(\mathbb{Z}^{\mathrm{h}G}) = 0,$$
since $\mathbb{Z}^{\mathrm{h}G}$ is in non-negative degrees. Thus, for an extension L/K of local fields with Galois group G, we have
$$L^\times/\mathrm{N}(L^\times) \simeq G^{\mathrm{ab}}.$$

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