20. Mechanical linkages and polynomial equations

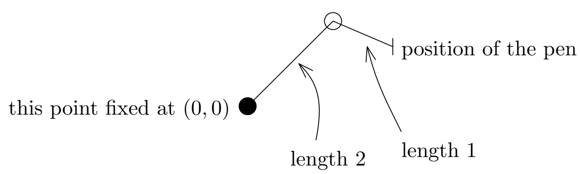
In this lecture, we explore other contexts which give rise to algebraic curves, even though one might not necessarily expect that.

- Mechanisms assembled from rods and joints often move along algebraic curves.
- To help us understand why that is the case, we introduce the resultant, a beautiful piece of classical algebra.

(20a) Linkages. Let's imagine a machine (the precise terminology is *mechanical linkage*) that consists of rods and rotary joints, all moving in a common plane; any intersections that may occur between the parts will be ignored. More precisely, we specify the lengths of the rods connecting various joints; certain joints will have a fixed positions, while the others can move freely, subject to the length constraints; and finally, somewhere on our machine, we place a pen. What are the possible positions of the pen? In other words, if we put a sheet of paper on our plane and let the machine move in all possible ways, what shape does the pen color?

Example 20.1. *Take the two-piece robot arm drawn below:*

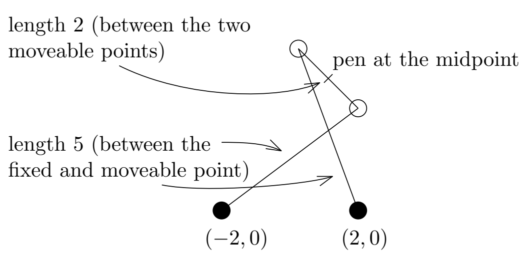
(20.1)



Clearly, the pen can reach any point in the region $\{1 \leq x^2 + y^2 \leq 9\}$.

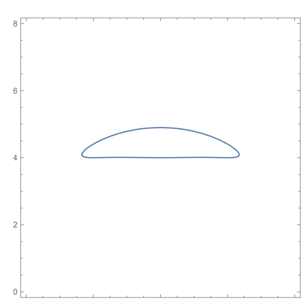
Example 20.2. *This is the Chebyshev mechanism, consisting of three rods with two fixed points:*

(20.2)



It draws a curve like this,

(20.3)



This was historically of interest in engineering because it has a piece which is almost (but not exactly) a straight line segment. It turns out that (20.3) *is part of the following algebraic curve:*

$$(x^2 + y^2)^3 - 8(7x^4 - 12x^2y^2 + 5y^4) + 16(49x^2 + 24y^2) = 0. \tag{20.4}$$

To be precise, the solution set of (20.4) *consists of the curve we've drawn, its mirror image under* $y \mapsto -y$ *(the same machine but in a position where the long rods hang downwards), and the (more mysterious) point* $(0, 0)$.

The naive idea is that if a linkage has "one degree of freedom", it draws part of an algebraic curve. Unfortunately, in this case the notion of "degree of freedom" has huge pitfalls, which every single engineering class discussing linkages steps into. The actual statement is this:

THEOREM 20.3. *The set S of all possible positions of the pen can be described by polynomial equalities and inequalities (one or several expressions $p(x, y) \,?\, 0$ with p polynomial, and $?$ being $=$, $\neq$, $>$, $\geq$, $<$, $\leq$). If, in that description, there is at least one equality, then S is a subset of an algebraic curve.*

EXAMPLE 20.4. *As we've already seen, the possible positions of the pen in* (20.1) *are described by the inequalities* $x^2 + y^2 \geq 1$ *and* $x^2 + y^2 \leq 9$.

EXAMPLE 20.5. *The possible positions of the pen in Chebyshev's mechanism is described by the equation* (20.4) *together with some inequality which excludes the point* $(0, 0)$, *let's say* $x^2 + y^2 > 0$.

(20b) Systems of polynomial equations. Let's see what the mathematics of a mechanical linkage looks like. Write (x_k, y_k) for the position of all the joints, and (x, y) for the position of the pen. Some joints are fixed, which means that the corresponding variables (x_k, y_k) are set to specific values. For the other (x_k, y_k), we have equalities of the form

$$(x_j - x_k)^2 + (y_j - y_k)^2 = \textit{squared length of the rod connecting the } j\textit{-th and } k\textit{-th joint.} \tag{20.5}$$

If the pen happens to be at the m-th joint, we have $x = x_m$ and $y = y_m$; in general, there will be a linear equation giving (x, y) in terms of the (x_k, y_k).

EXAMPLE 20.6. *For the Chebyshev mechanism* (20.2), *let* (x_1, y_1), (x_2, y_2) *be the positions of the moveable joints, and* (x, y) *that of the pen. The constraints are*

$$\begin{aligned} &(x_1 - 2)^2 + y_1^2 = 5^2, \\ &(x_2 + 2)^2 + y_2^2 = 5^2, \\ &(x_1 - x_2)^2 + (y_1 - y_2)^2 = 2^2, \\ &x = \tfrac{1}{2}(x_1 + x_2), \\ &y = \tfrac{1}{2}(y_1 + y_2). \end{aligned} \tag{20.6}$$

We want to describe those (x, y) *such that there exist* (x_1, y_1, x_2, y_2) *satisfying these equations.*

Changing notation a bit, the general situation is that we have variables $(x, y, z_1, \dots, z_n)$ (the z's are our previous x_k and y_k) and polynomial equations

$$
\begin{aligned}
&f_1(x, y, z_1, \dots, z_n) = 0,\\
&f_2(x, y, z_1, \dots, z_n) = 0,\\
&\dots\\
&f_m(x, y, z_1, \dots, z_n) = 0.
\end{aligned}
\tag{20.7}
$$

Then, we look at

$$S = \{(x, y) \in \mathbb{R}^2 \ : \ \text{there exist } (z_1, \dots, z_n) \text{ such that } (x, y, z_1, \dots, z_n) \text{ satisfies (20.7)}\}. \tag{20.8}$$

The question is, how does one get from such a description to polynomial equalities (or inequalities) that involve only (x, y)? In general, this is answered by the Tarski-Seidenberg theorem, which is beyond our scope; but we can get some idea of the mathematics involved.

(20c) The resultant. Take two polynomials is one variable z, of degree m and n:

$$
\begin{aligned}
f(z) &= a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0,\\
g(z) &= b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0.
\end{aligned}
\tag{20.9}
$$

The resultant $\mathrm{res}_z(f, g)$ is a number which depends on the coefficients of f and g, and which has the following very nifty property:

Lemma 20.7. *If f and g share a root, then* $\mathrm{res}_z(f, g) = 0$. *(More precisely, the resultant is zero exactly when f and g have a common factor. The common root case is when that factor is $(z - c)$ for some $c \in \mathbb{R}$.)*

The definition of the resultant is elementary but gruesome, as the determinant of a matrix of size $m + n$. The first n rows contain the coefficients of f padded with $n - 1$ zeros, and the last m rows do the same for g padded with $m - 1$ zeros:

$$
\mathrm{res}_z(f, g) = \det \begin{pmatrix}
a_m & a_{m-1} & \dots & a_2 & a_1 & a_0 & 0 & 0 & \dots & 0\\
0 & a_m & a_{m-1} & \dots & a_2 & a_1 & a_0 & 0 & \dots & 0\\
 & & \dots & & & & & & & \\
0 & \dots & 0 & 0 & a_m & a_{m-1} & \dots & a_2 & a_1 & a_0\\
\hdashline
b_n & b_{n-1} & \dots & b_1 & b_0 & 0 & 0 & 0 & \dots & 0\\
0 & b_n & b_{n-1} & \dots & b_1 & b_0 & 0 & 0 & \dots & 0\\
 & & \dots & & & & & & & \\
0 & \dots & 0 & 0 & 0 & b_n & b_{n-1} & \dots & b_1 & b_0
\end{pmatrix}
\tag{20.10}
$$

What does this have to do with our discussion? Take a simple of case of (20.8), which is that we have two polynomial equations in three variables,

$$
\begin{aligned}
f(x, y, z) &= 0,\\
g(x, y, z) &= 0
\end{aligned}
\tag{20.11}
$$

and look at

$$S = \{(x, y) \in \mathbb{R}^2 \ : \ \text{there exists a } z \in \mathbb{R} \text{ such that (20.11) holds}\}. \tag{20.12}$$

Pick some $(x, y) \in \mathbb{R}^2$, and insert those in f and g, so that they just become polynomials in z. If there exists a z such that $f(x, y, z) = 0$ and $g(x, y, z) = 0$, then those polynomials have a common root. Therefore,

$$S \subset \{(x, y) \ : \ \mathrm{res}_z(f, g) = 0\}. \tag{20.13}$$

It is important to clarify the situation here: when forming the resultant with respect to z, we write

$$\begin{aligned} f(x, y, z) &= a_m(x, y)z^m + a_{m-1}(x, y)z^{m-1} + \cdots + a_1(x, y)z + a_0(x, y), \\ g(x, y, z) &= b_n(x, y)z^n + b_{n-1}(x, y)z^{n-1} + \cdots + b_1(x, y)z + b_0(x, y). \end{aligned} \tag{20.14}$$

and then stick the $a_k(x, y)$ and $b_k(x, y)$ (themselves polynomials in x and y) into (20.10), so that $\mathrm{res}_z(f, g)$ is a function of (x, y). In fact, the formula (Cramer's formula for the determinant) shows that it is a polynomial!

Corollary 20.8. *If the polynomial $h(x, y) = \mathrm{res}_z(f, g)$ is nonzero, then the set S from* (20.12) *is a subset of the algebraic curve $C = \{h(x, y) = 0\}$.*

If this applies, it reduces us from two polynomial equations in three variables (20.11) to one equation in two variables ($h(x, y) = 0$).