1. Cutting and pasting polygons

Suppose that we have two polygonal shapes, but equal area. How can we prove that they have the same area? The simplest way would be to cut one into finitely many polygonal pieces, then reassemble those into the second shape.

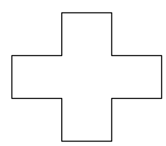
- We look at several examples such cut-and-paste arguments.
- Inspired by that, we ask whether every equality of areas can be proved in this way.

This is a way of approaching the notion of area (for polygons), without integrals or any kind of limit process.

(1a) Exploring the problem. Let's look at how cutting and pasting polygons can be fun.

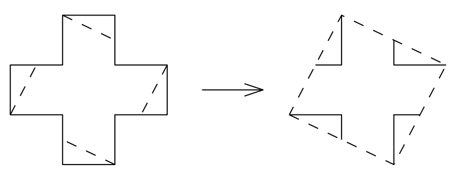
Example 1.1. *Take a Greek cross:*

(1.1)



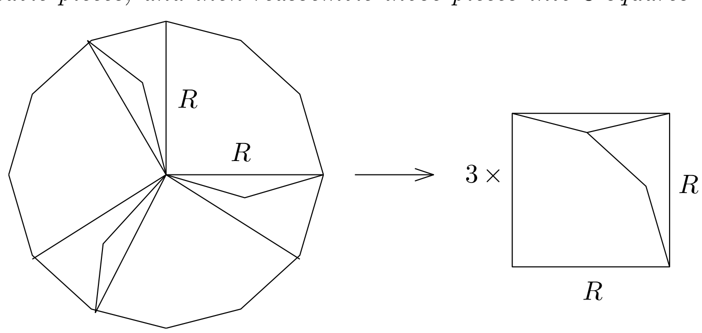
I would like to cut-and-paste transform it into a square. Let's say that each edge (edges is another word for sides) of the cross has length 1*. Then, its area is* 5*. Hence, a square of equal area must have side-lengths* $\sqrt{5}$*, which suggests the hypothenuse of a right-angled triangle with other side-lengths* 1 *and* 2*. Motivated by that, we find a solution:*

(1.2)



Example 1.2. *Take a regular* 12*-gon inscribed in a circle of radius* R*. Its area is exactly* $3R^2$*. One can prove that using trigonometric functions, but that's too complicated. Instead, we cut the* 12*-gon into* 9 *suitable pieces, and then reassemble those pieces into* 3 *squares of side-length* R*:*

(1.3)

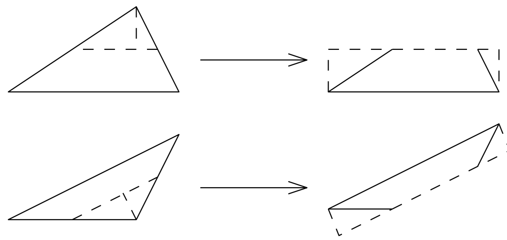


One has to figure out what cuts to make exactly, and why the pieces fit together: don't take my word for it, do it yourself!

Let's look for cut-and-paste strategies which could be useful in general.

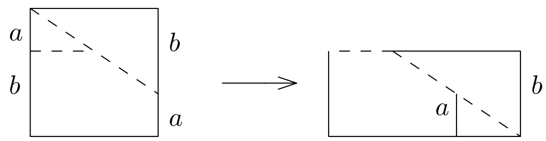
Strategy 1.3. *Cut-and-paste transforming a triangle into a rectangle is always possible:*

(1.4)



Strategy 1.4. *Cut-and-paste transforming a rectangle into another (of the same area, of course):*

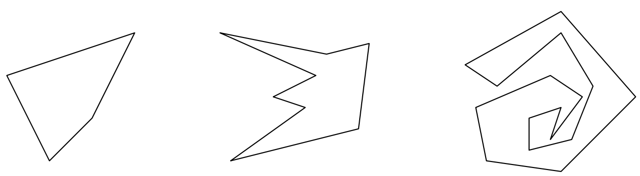
(1.5)



For this to work as drawn, we need $a \leq b$. Therefore, one such step can at most halve the height; we may have to repeat the process several times.

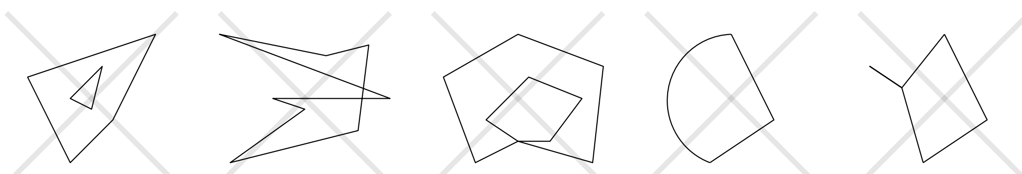
(1b) The general result. To turn this into a systematic discussion, we need to agree on what we are talking about. We only ever look at polygons in the plane. These *are* polygons:

(1.6)



These are *not* polygons:

(1.7)



The first example in (1.6) is a convex polygon. Convexity is not necessary for a polygon (the other two examples in (1.6) are not convex), but it's an important enough notion for us to take a short detour and define it properly.

Definition 1.5. *A polygon is* convex *if, when you take any two points on its boundary, the line segment connecting them never leaves the polygon.*

Intuitively, if you build a room in a convex shape, you can see from any point on the wall to any other point (it then also follows that any two people in the room can see each other). Let's get back to our main discussion:

Definition 1.6. *Two polygons are called* scissors congruent *if one of them can be cut into finitely many polygonal pieces, which can be moved around and reassembled to form the other polygon. "Moving around" consists of arbitrary Euclidean transformation (congruence transformations).*

Clearly, if two polygons are scissors congruent, then they must have the same area. We're interested in the converse, which is not a priori clear, but true:

THEOREM 1.7. *If two polygons have the same area, they are scissors congruent.*

The proof works as follows. Take the first polygon.

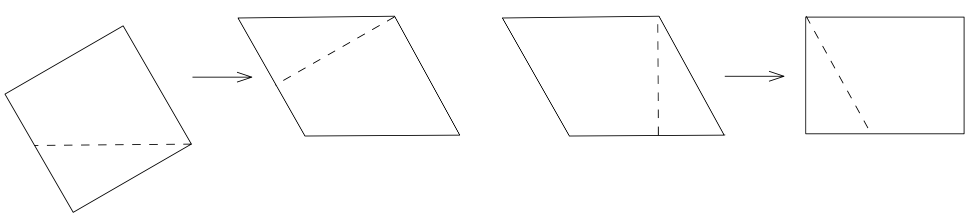
Step 1: Cut it into triangles. This is always possible.
Step 2: Apply Strategy 1.3 to each triangle, to transform it into a rectangle.
Step 3: Apply Strategy 1.4 to each rectangle, to make it size $1 \times$ (*something*).
Step 4: Take all those rectangles and paste them together into a single one of size $1 \times$(*something*).

We can apply the same process to the second polygon, and arrive at the same rectangle in the end (because that depends only on the area). The concluding idea is to run that process in reverse, so as to transform the rectangle into the second polygon.

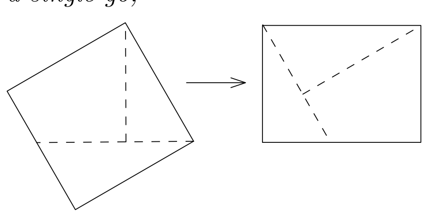
(1c) Using less transformations. Each Euclidean transformation is a combination of translations, rotations, and reflections. At no point in the argument above have we actually used reflections: so, Theorem 1.7 is true even if we only allow translations and rotations. Let's see if we can pare down the repertoire of transformations even more.

Fact 1.4 used only translations (no rotations), while Fact 1.3 additionally used 180° rotations. In the proof of Theorem 1.7, we can get up (3) by using only those two kinds of transformations. At that stage, we'll end up with a bunch of rectangles whose sides point in all sorts of directions. In the original Step 4, we (implicitly) rotated the rectangles to align their sides. We can avoid that by inserting another Step $2\frac{1}{2}$, which makes all rectangles into axis-parallel ones (with horizontal and vertical sides), *without rotating the pieces:*

STRATEGY 1.8. *Take a rectangle. One can apply a cut-and-paste process to it, which involves only translations, and whose outcome is another rectangle, rotated by any desired angle from the original one. For that, it is maybe simplest to first transform the rectangle into a square, using Strategy 1.4 (which satisfies our condition of using only translations). After that, one does the following:*

(1.8)


or, if you prefer to draw it in a single go,

(1.9)


Corollary 1.9. *Theorem 1.7 is still true if we restrict the notion of scissors congruence to using only translations and* 180 *degree rotations (instead of all Euclidean transformations).*

(1d) Using only translations. Getting bolder, we ask: how about getting away with no rotations at all, meaning using only translations? This is impossible in general. The obstacle is called Hadwiger invariants. Take a polygon P and a nonzero vector w in $\mathbb{R}^2$ (actually, we only need the direction of that vector, meaning that it doesn't matter if we multiply w by a positive number). The *Hadwiger invariant* $\mathrm{had}_w(P)$ is defined by

$$\mathrm{had}_w(P) = \sum_e \pm \mathrm{length}(e). \tag{1.10}$$

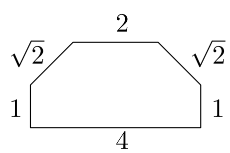
Here, e are edges (sides) of our polygon which are perpendicular (orthogonal) to w. We set the sign equal to $+$ if w points outwards along e, and $-$ if it points inwards. This definition means that we always have

$$\mathrm{had}_{-w}(P) = -\mathrm{had}_w(P). \tag{1.11}$$

If the polygon has n edges, there can only be at most $2n$ directions with nonzero Hadwiger invariants (namely, the pair of opposite directions perpendicular to each edge). However, there can be less than that, due to cancellations.

Example 1.10. *The hexagon below has* 6 *nonzero Hadwiger invariants. Here's one direction out of each pair, plus another direction where the Hadwidger invariant is zero due to cancellations:*

(1.12)



w	$\rightarrow$	$\nearrow$	$\uparrow$	$\nwarrow$
$\mathrm{had}_w(P)$	0	$\sqrt{2}$	-2	$\sqrt{2}$

The nifty choice of signs in (1.10) ensures that, if I cut P into P_1, P_2 by a straight-line cut, then

$$\mathrm{had}_w(P) = \mathrm{had}_w(P_1) + \mathrm{had}_w(P_2) \ \text{ for all } w. \tag{1.13}$$

The cut produces a new edge. That new edge occurs for P_1 and for P_2, but if a perpendicular direction to that edge points inwards for P_1, then it points outwards for P_2, hence the associated signs are opposite; which explains (1.13). Moreover, the Hadwiger invariants stay the same under translation. Therefore:

Theorem 1.11. *If two polygons P_1 and P_2 are scissors congruent in a way which uses only translations (and no other Euclidean transformations), then their Hadwiger invariants must agree:*

$$\mathrm{had}_w(P_1) = \mathrm{had}_w(P_2) \ \textit{ for all } w. \tag{1.14}$$

This means that besides equality of area, there are other conditions (possibly, one for every edge of P_1 or P_2) that need to be satisfied, if we are determined to use only translations.

Remark 1.12. *The analogue of Theorem 1.7 for three-dimensional polytopes is false. Again, this is because there are additional geometric quantities (not Hadwiger invariants, but the so-called Dehn invariant, whose definition is a little more tricky) which are additive under cut-and-paste, and unchanged under Euclidean transformations.*