10. An extremal characterization

Resonance frequencies and resonance modes are visibly akin to eigenvalues and eigenvectors. It's time to make use of this relation.

- Based on an idea from linear algebra, we explain an easy way to get an upper bound on the principal frequency from any choice of "test function". One can refine this by using several test functions, and the bounds become pretty good!
- Besides the practical computational aspect, this approach also establishes a number of interesting properties of the principal frequency.

(10a) The Rayleigh quotient. An $n \times n$ matrix A is called symmetric if $A_{ij} = A_{ji}$. In other words, A is equal to its transpose A^t. Given such a matrix, we look at the Rayleigh quotient

$$\frac{w \cdot Aw}{w \cdot w}, \quad w \neq 0. \tag{10.1}$$

If we restrict to vectors of length one, then this is just $w \cdot Aw$. The quotient extends it to all nonzero vectors, so that it has the same value on any multiple of w. One can think of it as a function on the sphere (two-dimensional sphere if $n = 3$), and as such, it has to achieve a minimum and maximum somewhere. What's important is that these values have meaning, in terms of the eigenvectors w of A, which are vectors that satisfy

$$Aw = \mu w \text{ for some } \mu \in \mathbb{R}, \text{ called the eigenvalue.} \tag{10.2}$$

THEOREM 10.1. *The minimal value of* (10.1) *is the lowest eigenvalue of A, and the vectors w that achieve that value are the corresponding eigenvectors. (There's a corresponding result for the maximum, but we won't be using it.)*

By turning the logic of the argument around, we can get upper bounds on the lowest eigenvalue:

COROLLARY 10.2. *Let μ be the smallest eigenvalue of A. Then*

$$\mu \leq \frac{w \cdot Aw}{w \cdot w} \quad \textit{for all nonzero vectors } w. \tag{10.3}$$

Moreover, if equality holds, then w is an eigenvector for μ.

Back to principal frequencies!

THEOREM 10.3. *Let λ be the principal frequency of a region U. Then*

$$\lambda^2 \leq \frac{\int_U \|\nabla f\|^2}{\int_U f^2} \tag{10.4}$$

for any function $f : U \to \mathbb{R}$ which is zero on the boundary of U, but not altogether zero. Equality holds exactly when f is the resonance mode corresponding to λ.

(10b) Applications. The theorem above is quite easy to use: you can stick in any f as a "test function", and get an upper bound for the principal frequency.

Example 10.4. *Look at the disc of radius* 1 *(centered at the origin). We can use any function* f *in* (10.4) *to get an upper bound for its principal frequency. Let's make a guess and take*

$$f(x,y) = 1 - x^2 - y^2, \tag{10.5}$$

which has $\|\nabla f\|^2 = 4(x^2+y^2)$*. We can compute the necessary integrals in radial coordinates,*

$$\int_U \|\nabla f\|^2 = \int_0^1 (2\pi r) 4r^2\, dr = 2\pi, \qquad \int_U f^2 = \int_0^1 (2\pi r)(1-r^2)^2\, dr = \pi/3. \tag{10.6}$$

We therefore get a bound $\lambda \leq \sqrt{6} = 2.449\ldots$ *for the principal frequency.*

The test function idea also has theoretical payoff:

Corollary 10.5. *Suppose that we have two regions with* $U \subset V$*. Then the principal frequency of* U *is greater than or equal to the principal frequency of* V*; meaning,* $\lambda_U \geq \lambda_V$*.*

To explain the argument, we need to clarify what test functions f can appear in Theorem 10.3: any function that's continuous on U, piecewise differentiable, and whose derivatives are continuous on each piece, is allowed. This allows us to integrate both f^2 and $\|\nabla f\|^2$, so the formula (10.4) makes sense. Now let's get back to $U \subset V$. We write λ_U and λ_V for their principal frequencies, and f_U for the principal mode of U. Let's extend f_U by zero over the rest of V, and call the result f_V. Of course, f_V is not the principal mode of V, but it does satisfy the conditions we've mention. Therefore,

$$\lambda_V^2 \leq \frac{\int_V \|\nabla f_V\|^2}{\int_V f_V^2} = \frac{\int_U \|\nabla f_U\|^2}{\int_U f_U^2} = \lambda_U, \tag{10.7}$$

and that's all there is to it.

Example 10.6. *Take again the disc of radius* 1*. It contains a square of side-length* $\sqrt{2}$*, and is contained in a square of side-length* 2*. We know that the principal frequency of an* l *by* l *square is* $\pi\sqrt{2}l^{-1}$*. For the principal frequency* λ *of the disc, we get*

$$\pi\sqrt{1/2} \leq \lambda \leq \pi. \tag{10.8}$$

The upper bound is much cruder than what we got from our previous test function, but the lower bound $\pi\sqrt{1/2} = 2.221\ldots$ *is new.*

(10c) Explaining the theorem. Take functions f and g on U. Green's theorem, applied to the vector field $(f\partial_y g, -f\partial_x g)$, says that

$$\int_U \partial_x(-f\partial_x g) - \partial_y(f\partial_y g) = \int_U -f\Delta g - \nabla f \cdot \nabla g = \text{some integral over the boundary of } U. \tag{10.9}$$

If f is zero on the boundary of U, then the boundary term becomes zero, which means

$$\int_U -f\Delta g = \int_U \nabla f \cdot \nabla g. \tag{10.10}$$

Using that, one can rewrite the quotient in the theorem above as

$$\frac{\int_U -f \cdot \Delta f}{\int_U f^2}. \tag{10.11}$$

If f is any resonance mode, then $\Delta f = -\lambda^2 f$ for the corresponding frequency λ, and so the quotient is just λ^2. This explains why, if we take f to be the principal resonance mode, we get the principal frequency. This, together with the linear algebra analogy (where $-\Delta$ plays the role of A), is as far as we'll get in explaining why Theorem 10.3 holds.

(10d) Using more than one test function. We can use any test function to get an upper bound on the principal frequency, but depending on how good we are at picking the function, the bound might be more or less useful. Here's a more systematic way of applying the idea. Choose functions $f_1, \dots, f_n$ on U, each of which is zero on the boundary. We are looking for a test function which is a linear combination of them,

$$f = w_1 f_1 + \cdots + w_n f_n, \quad \text{where } w = (w_1, \dots, w_n) \in \mathbb{R}^n \text{ can be any nonzero vector.} \tag{10.12}$$

For the fundamental frequency, we get

$$\lambda^2 \leq \frac{\int_U \|\nabla f\|^2}{\int_U f^2} = \frac{\sum_{i,j=1}^n A_{ij} w_i w_j}{\sum_{i,j=1}^n B_{ij} w_i w_j} = \frac{w \cdot Aw}{w \cdot Bw}, \tag{10.13}$$

where

$$A_{ij} = \int_U \nabla f_i \cdot \nabla f_j, \quad B_{ij} = \int_U f_i f_j. \tag{10.14}$$

Our job is then to pick w so that the quotient (10.13) becomes as small as possible, so that we get the best bound for λ^2. Luckily, there's a linear algebra theorem for that, which generalizes the one from the start of the lecture:

THEOREM 10.7. *Let A and B be symmetric matrices of size n. We also require that B is positive, in the sense that $w \cdot Bw > 0$ for all nonzero $w \in \mathbb{R}^n$. Then, the minimum of the quantity*

$$\frac{w \cdot Aw}{w \cdot Bw}, \quad w \in \mathbb{R}^n \text{ nonzero}, \tag{10.15}$$

is the lowest eigenvalue of $B^{-1}A$.

In case you don't like eigenvectors, the eigenvalues of $B^{-1}A$ are also the roots of the polynomial $p(t) = \det(tB - A)$. Let's summarize the outcome:

COROLLARY 10.8. *Pick functions $f_1, \dots, f_n$ on U, each of which is zero on the boundary, and compute the associated matrices A and B. Then the principal frequency λ satisfies*

$$\lambda^2 \leq \text{any root of } p(t) = \det(tB - A). \tag{10.16}$$

EXAMPLE 10.9. *Let's return to the disc, with the repertoire of functions $f_1 = 1 - x^2 - y^2$, $f_2 = f_1^2$, $f_3 = f_1^3$. After computing all the required integrals and the determinant (all of which we skip here), one gets*

$$\det(tB - A) = \pi^3(t^3/378000 - t^2/2520 + 2t/175 - 4/75), \tag{10.17}$$

which has its smallest root at $t = 5.783\dots$. This gives us a bound $\lambda \leq \sqrt{t} = 2.404\dots$, a big improvement over Example 10.4 (the four digits we've written down actually agree with the value of λ, computed numerically by other means).