11. Symmetrization

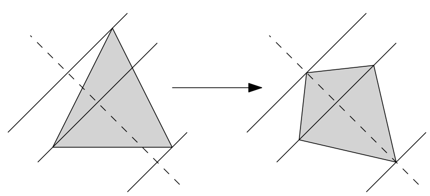
Symmetrization is a process that takes a region and some straight line (the axis), and produces a new region that's symmetric with respect to that axis.

- We define symmetrization, and explore its effect on area and perimeter. This is elementary geometry!
- We discuss some examples, including an in-depth look at symmetrization of triangles.
- Finally, returning to the principal frequency, symmetrization provides inequalities between the principal frequencies of different regions.

(11a) Definition and first properties. Fix a line l in the plane. Given a region U, symmetrization with respect to the axis l works as follows. Take any line $l^\perp$ perpendicular to l, look at the total length of $U \cap l^\perp$, and draw the open interval of the same length inside $l^\perp$ centered on the point $l \cap l^\perp$. The union of those intervals, for all $l^\perp$, is the symmetrization of U, written as $S_l(U)$. It is a new region, symmetric with respect to l. The easiest situation is when U is convex, in which case $U \cap l^\perp$ consists of a single interval. Then, we just slide the interval along $l^\perp$ until it becomes symmetric with respect to l, and apply the same to all $l^\perp$ to form $S_l(U)$. In the non-convex case, one may have to merge several intervals into one, but the idea is the same.

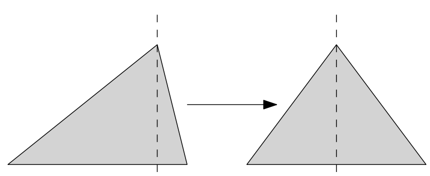
Example 11.1. *Take a triangle. Mostly, if we symmetrize it, we get a kite (a quadrilateral which is symmetric with respect to reflection along one diagonal):*

(11.1)

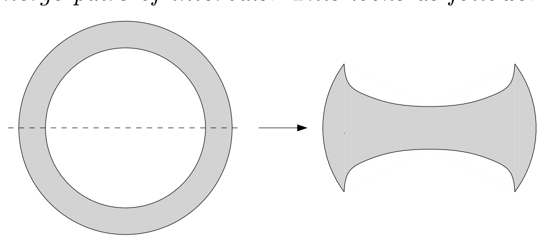


The exception to "mostly" is symmetrization with respect to an altitude, which gives another triangle (necessarily an isosceles one, because of the symmetry):

(11.2)



Example 11.2. *Take a circular strip, lying between two concentric circles. In this case, when we symmetrize, we have to merge pairs of intervals. This looks as follows:*



(11.3)

In formulae, we would have

$$
\begin{aligned}
&U = \{c \leq \sqrt{x^2+y^2} \leq d\}, \\
&S_l(U) = \{c \leq |x| \leq d,\ |y| \leq \sqrt{d^2-x^2}\} \cup \{|x| \leq c,\ |y| \leq \sqrt{d^2-x^2} - \sqrt{c^2-x^2}\}.
\end{aligned}
\tag{11.4}
$$

LEMMA 11.3. *The area of $S_l(U)$ is the same as that of U.*

This is due to the definition of area as integral of the function that gives the lengths of $l \cap l^{\perp}$ (Cavalieri's principle of indivisibles if you're historically minded, or Fubini's theorem for the analytically informed).

THEOREM 11.4. *The perimeter of $S_l(U)$ is less or equal than the perimeter of U.*

Let's look at the situation (not the most general one, but close enough) familiar from calculus, where U is the region between two graphs $f(x)$ and $g(x)$,

$$U = \{a \leq x \leq b,\ \ f(x) \leq y \leq g(x)\}. \tag{11.5}$$

Here, we're assuming $f(x) \leq g(x)$, and that $f(a) = g(a)$, $f(b) = g(b)$. The perimeter of U is then just the sum of the lengths of the two graphs,

$$\int_a^b \sqrt{1+f'(x)^2} + \sqrt{1+g'(x)^2}\ dx. \tag{11.6}$$

Let's symmetrize with respect to the x-axis,

$$S_l(U) = \{a \leq x \leq b,\ \ |y| \leq \tfrac{1}{2}(g(x)-f(x))\}. \tag{11.7}$$

This is the region between the graph of $\frac{1}{2}(f(x)-g(x))$ and $\frac{1}{2}(g(x)-f(x))$, so we have a similar formula for the perimeter,

$$\int_a^b 2\sqrt{1+\tfrac{1}{4}(g'(x)-f'(x))^2}\ dx. \tag{11.8}$$

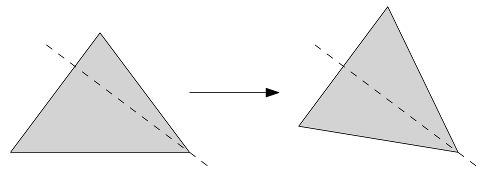
The claim is that the integrand here is less or equal than that in the previous formula. If we write $F = f'(x)$, $G = g'(x)$, then what we need is

$$2\sqrt{1+(G-F)^2/4} \leq \sqrt{1+F^2} + \sqrt{1+G^2}. \tag{11.9}$$

This inequality holds for all numbers F and G. One can prove it by squaring repeatedly and cleaning up terms, which we'll leave to you.

(11b) Isosceles triangles. As discussed before, if we take a triangle and symmetrize it with respect to an altitude, we get another triangle, which is always isosceles. What if we already start with an isosceles triangle, let's call it T? One of its altitudes is the axis of symmetry, so symmetrizing with respect to that changes nothing. Let's symmetrize with respect to one of the two other altitudes, and call this process *altitude symmetrization* $S(T)$. (Up to congruence, it doesn't matter which of the two altitudes you pick.)

(11.10)



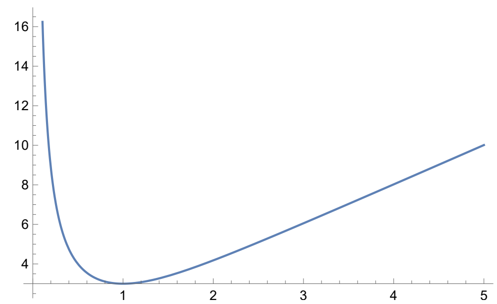
Lemma 11.5. *Take an isosceles triangle which is not equilateral. If we apply altitude symmetrization to it, then the new triangle has smaller perimeter than the old one.*

This is an elementary geometry exercise, which we'll skip. Let's look generally at the perimeters of isosceles triangles. For simplicity, we use triangles with area $\sqrt{3}/4$, so that the equilateral one has side-length 1. An isosceles triangle with are $\sqrt{3}/4$ and base b has height h (measured with respect to the base) and perimeter

$$p = b + 2\sqrt{(b/2)^2 + h^2} = b + \sqrt{b^2 + 4h^2} = b\Big(1 + \sqrt{1 + \frac{3}{b^4}}\Big). \tag{11.11}$$

Let's look at the function $p = p(b)$:

(11.12)



It has an absolute minimum $p(1) = 3$ corresponding to the equilateral triangle. For every $p > 3$, there are exactly two values b with $p(b) = p$. This has the following consequence:

Theorem 11.6. *Start with any isosceles triangle, and apply altitude symmetrization over and over. The result is a sequence of isosceles triangles, which either turns equilateral after finitely many steps, or else becomes closer and closer to equilateral in the limit.*

Proof. As before, we just discuss triangles with area $\sqrt{3}/4$. One can scale the discussion up and down to any area. As the theorem says, it's an option to get an equilateral triangle after finitely many steps. If that's not the case, then by our previous Lemma, we get a sequence of triangles with bases $b_1, b_2, b_3, \dots > 0$ and perimeters $p_1, p_2, p_3, \dots > 3$, with

$$p_1 > p_2 > p_3 > \cdots \tag{11.13}$$

If the limit of those p_n is 3, then by looking at the graph of the function, we see that the limit of the b_n must be 1, so we are converging to an equilateral triangle. What if the limit of the p_n is some number $p > 3$? To that number correspond two values of b, meaning two isosceles triangles. Because of the limiting process, at least one of those two must have the property that its perimeter doesn't decrease under isosceles symmetrization, which contradicts our Lemma. So $p > 3$ is after all impossible! □

(11c) Back to the principal frequency. With respect to symmetrization, the principal frequency behaves like the perimeter:

Theorem 11.7. *The principal frequency of $S_l(U)$ is less or equal than that of U.*

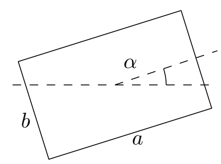
The theorem is proved using the minimizing idea from the previous lecture: one takes a test function f on U, and produces another function $S_l(f)$ on $S_l(U)$, such that

$$\int_{S_l(U)} S_l(f)^2 = \int_U f^2, \quad \int_{S_l(U)} \|\nabla S_l(f)\|^2 \leq \int_U \|\nabla f\|^2. \tag{11.14}$$

Obviously, the gap in this explanation is the definition of $S_l(f)$, and why it has those properties. There are geometric ways to understand this, which are quite similar to our discussion of the behaviour of the perimeter, but we prefer not to get into that discussion here. It's the consequences which make this fact interesting.

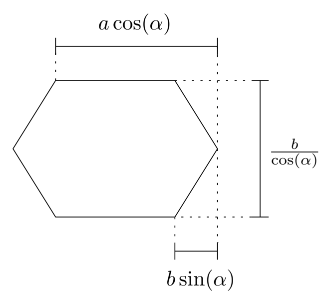
EXAMPLE 11.8. *Take an $a \times b$ rectangle, rotated by some angle α. We assume that the angle of rotation is small enough so that the x-axis still passes through the b-side, and symmetrize with respect to that axis.*

(11.15)



The outcome is a hexagon symmetric with respect to the x-axis, and also with respect to rotation by $180°$. It has the following measurements:

(11.16)



Therefore, if we take a, b and α so that $b\sin(\alpha) = \frac{1}{2}$, $b/\cos(\alpha) = \sqrt{3}$, $a\cos(\alpha) = \frac{3}{2}$, the symmetrized shape is a regular hexagon with side-length 1. The quotient of the first two equations says that $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = 1/\sqrt{3}$, and with that at hand, one can calculate everything. For the fundamental frequency of the hexagon, this yields a reasonably good bound (much better than what one gets from putting a rectangle inside the hexagon)

$$\lambda \leq \pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{2\pi}{3}\sqrt{5 - \frac{4}{3}\sqrt{6}} = 2.75794\ldots \tag{11.17}$$

As a more serious application, suppose that we start with a triangle. By repeated symmetrization as in our previous discussion, we can turn it into a triangle which is either equilateral or very very close to it. In the latter case it contains a slightly smaller equilateral triangle, and is contained in a slightly larger equilateral one, so the fundamental frequency is very close to that of the equilateral one, and we can actually make the error as small as we want. So we've proved this:

COROLLARY 11.9. *Among all triangles with a given area, the equilateral one achieves the minimum of the principal frequency.*

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