12. Smooth loops

We look at smoothly curved loops in the plane. This is the curvy analogue of our previous study of polygonal loops. The basic ideas are similar, but the techniques involved in realizing them are different.

- We define the winding number of a smooth loop around a point, by an integral formula.
- We discuss properties of winding numbers, with emphasis on deformation invariance.

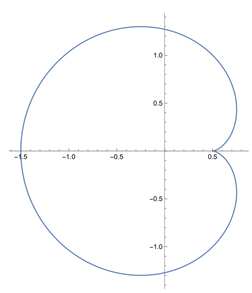
(12a) Definition. Take functions $x(t)$ and $y(t)$, defined for $t \in \mathbb{R}$ and which have derivatives of all orders (these are usually called smooth functions), together with a $T > 0$ such that both functions are T-periodic:

$$x(t+T) = x(t),\ y(t+T) = y(t) \quad \text{for all } t. \tag{12.1}$$

We then call the parametrized curve $c(t) = (x(t), y(t)) \in \mathbb{R}^2$ a smooth loop. Note that the choice of T is part of the definition of a smooth loop: for instance, $c(t) = (\cos(t), \sin(t))$, we could take $T = 2\pi$ (loop goes once around the circle) or $T = 4\pi$ (loop goes twice around the circle).

Example 12.1. *Even though the loop is smooth as a parametrized curve, the shape in the plane it traces out can appear non-smooth, at those points where the speed of $c(t)$ becomes zero. For instance, take $x(t) = \cos(t) - \cos(2t)/2$, $y(t) = \sin(t) - \sin(2t)/2$. With $T = 2\pi$, this is a smooth loop, and looks like this:*

(12.2)



At the point $t = 0$, we have $x'(t) = -\sin(t) + \sin(2t) = 0$ and $y'(t) = -\cos(t) + \cos(2t) = 0$, and that's where the kink happens.

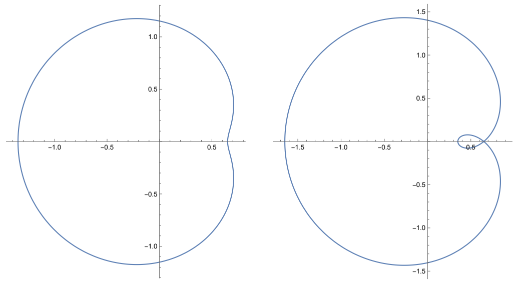
Take two smooth loops $c_0(t)$ and $c_1(t)$, and let's say that they both have period T. A *deformation of c_0 into c_1* is given by $c_s(t) = (x(s,t), y(s,t))$, for $s \in [0,1]$ and $t \in \mathbb{R}$, such that $x(s,t)$ and $y(s,t)$ are arbitrarily often differentiable, and the periodicity condition is preserved throughout:

$$x(s, t+T) = x(s,t),\ y(s,t+T) = y(s,t). \tag{12.3}$$

Here, $s \in [0,1]$ is an auxiliary parameter; for $s = 0, 1$ we get our original two loops, and c_s for general s yields a family of smooth loops which gradually interpolate between them.

Example 12.2. *Even though the deformation is smooth, the loops can appear to change shape. For example, take $x(s,t) = \cos(t) - s\cos(2t)$, $y(s,t) = \sin(t) - s\sin(2t)$, which is certainly a* *smooth deformation. For $s = 1/2$ that gives the previous example. When we cross that parameter value, a small curl appears in the loop:*

(12.4)



$s = 0.35$ $\qquad$ $s = 0.65$

FACT 12.3. *Any two smooth loops (with the same T) can be deformed into each other.*

One simply moves from $c_0(t)$ to $c_1(t)$ by a straight line segment, which means setting

$$c_s(t) = c_0(t) + s(c_1(t) - c_0(t)) = (1-s)c_0(t) + sc_1(t). \tag{12.5}$$

This seems to say that deformation isn't very interesting. Which is true: that notion only becomes worth while talking about if we put additional restrictions on what deformations are allowed.

Nrom now, we'll gradually dinish our use of "smooth"; all loops that occur are intended to be smooth ones, so this word isn't really necessary.

(12b) Winding numbers. Let c be a smooth loop, and $q \in \mathbb{R}^2$ a point not lying on that loop. The winding number of c around q is defined as

$$\mathrm{wind}(c,q) = \frac{1}{2\pi}\int_0^T \frac{(c(t)-q)\times c'(t)}{\|c(t)-q\|^2}\,dt. \tag{12.6}$$

One can write the formula more symmetrically as

$$\mathrm{wind}(c,q) = \frac{1}{2\pi}\int_0^T \frac{c(t)-q}{\|c(t)-q\|} \times \frac{d}{dt}\Big(\frac{c(t)-q}{\|c(t)-q\|}\Big)\,dt. \tag{12.7}$$

It may look as if this can't possibly equivalent to the previous expression, because

$$\frac{d}{dt}\frac{c(t)-q}{\|c(t)-q\|} = \frac{c'(t)}{\|c(t)-q\|} + (c(t)-q)\,\frac{d}{dt}\Big(\frac{1}{\|c(t)-q\|}\Big), \tag{12.8}$$

and the second term has no counterpart in (12.6). The answer to that quandary is that the term in question is a scalar multiple of $c(t) - q$, hence contributes zero if we take the cross product with that vector.

To see what the integral formula means geometrically, let's write a smooth loop in polar coordinates centered at q,

$$c(t) = q + r(t)(\cos\theta(t), \sin\theta(t)). \tag{12.9}$$

Then $(c(t) - q)/\|c(t) - q\| = (\cos\theta(t), \sin\theta(t))$, and the winding number integral is

$$\frac{1}{2\pi}\int_0^T \begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix} \times \begin{pmatrix}-\sin\theta(t)\\ \cos\theta(t)\end{pmatrix}\theta'(t)\,dt = \frac{1}{2\pi}\int_0^T \theta'(t)\,dt = \frac{1}{2\pi}\big(\theta(T)-\theta(0)\big). \tag{12.10}$$

When we write (12.9), we want $r(t)$ and $\theta(t)$ to vary continuously with t. That requirement may force us to choose $\theta(t)$ not to be periodic: from the ambiguity of polar coordinates, we know that the values $\theta(t)$ and $\theta(t+T)$ can differ by an integer multiple of 2π. Our computation shows that this multiple is the winding number, as defined by the integral formula:

(12.11) $$\theta(t+T) = \theta(t) + 2\pi \,\mathrm{wind}(c, q).$$

This shows that the integral is always an integer. More importantly, it exactly reproduces the original intuition of counting the number of turns we have to do, while standing at q and looking towards the point $c(t)$ as it moves around the loop.

(12c) Properties of the winding number. The computational techniques that we have learned in the polygonal case carry over to smooth loops. For instance, the ray-cutting formula works as follows. Take a ray going from q to infinity, in direction w, such that:

(12.12) Wherever that ray meets $c(t)$, the vectors w and $c'(t)$ are linearly independent.

(This means that the ray crosses our loop transversally.) Then,

(12.13) $$\mathrm{wind}(c, q) = \sum_{\substack{\text{those } t \in [0,T) \text{ such} \\ \text{that } c(t) \text{ lies on the ray}}} \mathrm{sign}(w \times c'(t)).$$

Example 12.4. *Look at* $c(t) = (\cos(5t), \sin(3t))$*, with* $T = 2\pi$*. We are interested in the winding number around the origin* o *(which our loop avoids). Take a horizontal ray going to the right, which means* $w = (1, 0)$*. This hits our loop at those times* t *such that*

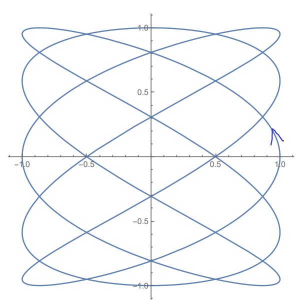
(12.14) $$\sin(3t) = 0, \quad \cos(5t) > 0.$$

In the interval $[0, 2\pi)$*, those times are* $t = 0$*,* $t = \pi/3$*,* $t = 5\pi/3$ *(because* $\cos(0) = 1$ *and* $\cos(5\pi/3) = \cos(25\pi/3) = \frac{1}{2}$*). We have*

(12.15) $$w \times c'(t) = (1,0) \times (-6\sin(5t), 3\cos(3t)) = 3\cos(3t) \begin{cases} 3 & t = 0, \\ -3 & t = \pi/3, \\ -3 & t = 5\pi/3. \end{cases}$$

Therefore $\mathrm{wind}(c, o) = 1 - 1 - 1 = -1$*. Of course, if we took a different ray, the number of intersection points could be different, but the total contribution would remain the same. Here's a picture of the loop so that you can check the computation visually:*

(12.16)



Proposition 12.5. *Take two smooth loops* c_0, c_1 *(with the same* T*) which avoid* q*. If they can be deformed into each other without ever passing through* q*, which means that all* c_s *avoid* q*, then* $\mathrm{wind}(c_0, q) = \mathrm{wind}(c_1, q)$*.*

We've seen the relevant kind of argument before: from the integral formula, one sees that $\mathrm{wind}(c_s, q)$ varies continuously with s. But we also know it's an integer, so it must be constant in s. Deformation invariance is frequently applied like this: the winding number doesn't change if we "wiggle the loop a little", since the original loop and the wiggled one can be joined by a deformation without crossing q. How much we're allowed to wiggle depends on how far the original loop was from the point q that we're computing the winding number for. Here is an explicit criterion:

COROLLARY 12.6. *(Man-dog-lamppost theorem) Suppose that c_0, c_1 are smooth loops (with the same period T), and q a point, such that*

$$\|c_1(t) - c_0(t)\| < \|c_0(t) - q\| \quad \textit{for all } t. \tag{12.17}$$

Then $\mathrm{wind}(c_0, q) = \mathrm{wind}(c_1, q)$.

Here, c_0 is the original loop, and c_1 is the wiggled one. To prove the equality of winding numbers, we use the straight-line deformation c_s, $0 \leq s \leq 1$, from (12.5). The important thing is that $c_s(t)$ never becomes equal to q. This is geometrically intuitive, or one can argue by contradiction:

$$\begin{aligned} c_s(t) = q \;\Rightarrow\; c_0(t) - q = s(c_1(t) - c_0(t)) \;\Rightarrow\; \|c_0(t) - q\| = s\|c_1(t) - c_0(t)\| \\ \Rightarrow\; \|c_0(t) - q\| \leq \|c_1(t) - c_0(t)\|, \quad \text{which is impossible.} \end{aligned} \tag{12.18}$$

EXAMPLE 12.7. *We'll compute the winding number of*

$$c_1(t) = (\cos(3t) + \cos(5t)/10, \sin(3t) + \sin(5t)/10), \quad (T = 2\pi), \tag{12.19}$$

around the origin o. Our loop can be seen as a slight wiggle on $c_0(t) = (\cos(3t), \sin(3t))$, which goes three times around a circle. Quantitatively, $\|c_0(t) - o\| = \|(\cos(3t), \sin(3t))\| = 1$, while $\|c_1(t) - c_0(t)\| = \|(\cos(5t)/10, \sin(5t)/10)\| = 1/10$. By man-dog-lamppost, $\mathrm{wind}(c_1, 0) = \mathrm{wind}(c_0, 0) = 3$.

The name can help you remember what's going on. The idea is that $c_0(t)$ is the position of a man walking in a smooth loop, around a lamppost q. The man is holding a dog on a leash: the position of the dog is $c_1(t)$, and the length of the leash is $\|c_1(t) - c_0(t)\|$. If the leash remains shorter than the distance from the man to the lamppost, which is $\|c_0(t) - q\|$, the leash can't get tangled around the lamppost. Therefore, after getting back to their starting positions, the man and the dog have circled the lamppost the same amount of times.

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