13. Equations in two variables

Maybe surprisingly, winding numbers can be used to prove existence results for solutions of systems of equations (two equations in two variables).

- Given such a system, and a prospective region where a solution could be located, one constructs an appropriate smooth loop. If that loop has nonzero winding number, the system must have at least one solution in our region. (This is similar to how one can use the intermediate value theorem to prove, for instance, that there is an $x \in (0, \pi/2)$ with $\cos(x) = x$.)
- Deformation methods for the winding number, such as man-dog-lamppost, are particularly useful in this context.

(13a) Existence of solutions. Let's start with functions $g(a,b)$ and $h(a,b)$, defined and smooth (have derivatives of any order) for all $(a,b) \in \mathbb{R}^2$. We are given (x,y), and are looking for solutions (a,b) of

(13.1)
$$\begin{aligned} g(a,b) &= x, \\ h(a,b) &= y, \end{aligned}$$

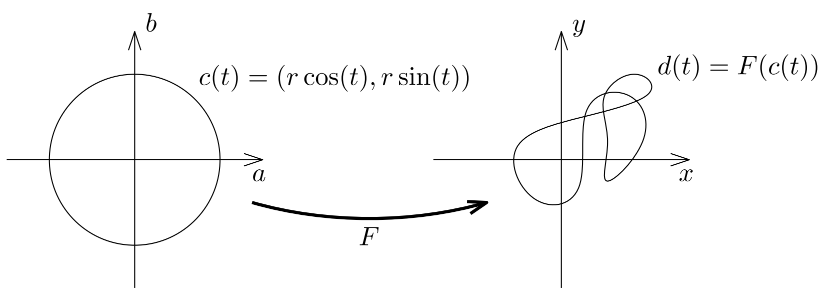
This is pretty general, g and h can be almost anything! For a more geometric picture, we combine our functions into a map

(13.2)
$$F(x,y) = (g(x,y), h(x,y)) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

In your mind, imagine two copies of the plane: that where F is defined, with coordinates (a,b), and that where it takes values, with coordinates (x,y). Then, if $q = (x,y)$ is given, what we are looking for in (13.1) are $p = (a,b)$ such that $F(p) = q$. Fix some $r > 0$, take the circle $c(t) = (r\cos(t), r\sin(t))$ of radius r around the origin (a loop with $T = 2\pi$), and look at its image under F:

$$d(t) = F(c(t)) = F(r\cos(t), r\sin(t)) = (g(r\cos(t), r\sin(t)), h(r\cos(t), r\sin(t))).$$

(13.3)



We'll be interested in the winding number of d around our chosen point q. Of course, for that to be defined, we have to assume that $d(t)$ never becomes equal to q: in other words, there shouldn't be any solutions of $F(p) = q$ on the circle $\|p\| = r$.

THEOREM 13.1. *Suppose that* $\mathrm{wind}(d,q) \neq 0$. *Then there must be a*

(13.4)
$$p \in \mathbb{R}^2 \textit{ with } \|p\| < r, \textit{ which solves } F(p) = q.$$

PROOF. Look at the deformation obtained by shrinking the circle in the (a, b) plane, depending on a parameter $s \in [0, 1]$:

$$\begin{aligned} c_s(t) &= (sr\cos(t), sr\sin(t)), \\ d_s(t) &= F(c_s(t)) = F(sr\cos(t), sr\sin(t)). \end{aligned} \tag{13.5}$$

At one end, $d_0(t) = F(0, 0)$ is the constant path. At the other end, $d_1(t) = d(t)$ is the path from our statement. The proof is by contradiction. Suppose that there is no solution (13.4). This implies that all loops d_s avoid q. By deformation invariance of the winding number, one would have $\mathrm{wind}(d_0, q) = \mathrm{wind}(d_1, q)$. But d_0 is a constant path, and therefore its winding numbers are 0, which is a contradiction. □

EXAMPLE 13.2. *We want to show that there's a solution $(a, b) \in \mathbb{R}^2$ of*

$$\begin{aligned} a - \cos(a + b^4) &= 0, \\ b - \cos(ab) &= 0, \end{aligned} \tag{13.6}$$

so $F(a, b) = (a - \cos(a + b^4), b - \cos(ab)) = (a, b) - (\cos(a + b^4), \cos(ab))$. The relevant loop is

$$d(t) = \big(r\cos(t), r\sin(t)\big) - \big(\cos(r\cos(t) + r^4\sin(t)^4), \cos(r^2\cos(t)\sin(t))\big). \tag{13.7}$$

To see whether our method applies, we need to know $\mathrm{wind}(d, o)$*, where $o = (0, 0)$ is the origin, and r has been chosen appropriately (we don't yet know how). Looking at* (13.7)*, the two terms have somewhat different sizes:*

$$\|(r\cos(t), r\sin(t))\| = r, \tag{13.8}$$

$$\|(\cos(r\cos(t) + r^4\sin(t)^4), \cos(r^2\cos(t)\sin(t)))\| < 2; \tag{13.9}$$

in the second case, this is because both the x and y coordinate lie in $[-1, 1]$. If we choose $r \geq 2$, the man-dog-lamppost theorem applies, with $(r\cos(t), r\sin(t))$ being the man, $d(t)$ the dog, and the lamppost at the origin $o = (0, 0)$. The consequence is that

$$\mathrm{wind}(d, o) = \mathrm{wind}(t \mapsto (r\cos(t), r\sin(t)), o) = 1. \tag{13.10}$$

It follows that (13.6) *has a solution with $a^2 + b^2 \leq 2^2 = 4$. Note that it's pretty clearly impossible to find the solution explicitly!*

Our argument didn't use anything about (13.6) except that one side was just (a, b), and the other side was bounded (13.9). In fact, the same reasoning gives a general statement:

COROLLARY 13.3. *Suppose that $k(a, b)$ and $l(a, b)$ are functions (defined on $\mathbb{R}^2$ and smooth) which are bounded (above and below, with bounds that hold for all a, b). Then, the system of equations*

$$\begin{aligned} a &= k(a, b), \\ b &= l(a, b) \end{aligned} \tag{13.11}$$

always has a solution.

(13b) More examples. So far, we have only dealt with cases where the winding number is 1. Let's enlarge our repertoire:

EXAMPLE 13.4. *Take $F(a,b) = (a^2-1, b)$, $q = (0,0)$, and $r > 1$. The relevant loop is*

$$d(t) = F(r\cos(t), r\sin(t)) = (r^2\cos(t)^2 - 1, r\sin(t)). \tag{13.12}$$

Let's compute the winding number using the intersect-a-ray approach. Specifically, we look at points where $d(t)$ is a positive multiple of $w = (1,0)$. This happens at $t = 0$ and $t = \pi$, where

$$d(0) = d(\pi) = (r^2 - 1, 0),\ d'(0) = (0, r),\ d'(\pi) = (0, -r). \tag{13.13}$$

Therefore, $d'(t) \times w$ is negative at $t = 0$ and positive at $t = \pi$, which means that the winding number is zero! This may be surprising because we clearly have solutions $(a,b) = (-1,0)$ and $(a,b) = (1,0)$ of $F(a,b) = (0,0)$. This is not a contradiction to our theorem, it just means that the converse implication doesn't hold in general.

(13c) A counting formula. You may have realized that in our context, two numbers appear: first, the winding number; and second, the number of solutions to our system of equations. The existence theorem says that if the first is nonzero, so is the second. That doesn't mean that the two numbers are equal: indeed, we've seen in examples that that's not the case generally; and moreover, such an equality is a priori impossible, as the winding number can be negative, and on the other hand the number of solutions can be infinite. Nevertheless, there is a relation, under certain additional assumptions:

THEOREM 13.5. *Look at a loop* (13.3). *Assume that for every p as in* (13.4)*, the partial derivatives $\partial F/\partial a$ and $\partial F/\partial b$, taken at the point $(a,b) = p$, are linearly independent vectors. Then*

$$\mathrm{wind}(d,q) = \sum_p \mathrm{sign}\Big(\frac{\partial F}{\partial a} \times \frac{\partial F}{\partial b}\Big). \tag{13.14}$$

Here, the sum is over all (13.4)*, and the partial derivatives are taken at those points.*

Let's revisit Example 13.4. We have

$$\frac{\partial F}{\partial a} \times \frac{\partial F}{\partial b} = (2a, 0) \times (0,1) = 2a, \tag{13.15}$$

which has opposite signs at $(a,b) = (1,0)$ and $(a,b) = (-1,0)$. Hence, the two contributions on the right hand side of (13.14) cancel each other out, which confirms our previous computation $\mathrm{wind}(d,o) = 0$.

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