14. Complex polynomials

Given how long we've been talking about the plane, it's surprising that complex numbers haven't appeared so far. We now fix this:

- One can use winding numbers to detect zeros of complex polynomials.
- Unlike the case of real equations, this is an "if and only if" process, and provides a count of how many zeros lie in a disc of radius r, assuming that the zeros are counted with their proper positive multiplicities.

(14a) Complex numbers. A complex number is given by its real and imaginary parts, $z = x + iy$, hence is the same as a point (x, y) in the plane. One writes $|z|$ instead of $\|z\|$ for its length, meaning

$$|z| = \sqrt{x^2 + y^2} \quad \text{for } z = x + iy. \tag{14.1}$$

There's a famous formula for trigonometric functions in terms of the complex exponential,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{14.2}$$

This means that complex numbers are written in radial coordinates as $z = re^{i\theta}$. As one sees from that, the product of complex numbers multiplies the radii and adds the angles:

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)e^{i(\theta_1+\theta_2)}. \tag{14.3}$$

One can think of smooth loops as taking values in complex numbers, meaning $c(t) \in \mathbb{C}$. The simplest example may be the loop

$$c(t) = e^{int}, \tag{14.4}$$

with $T = 2\pi$, for some integer n. This goes n times around the radius 1 circle (if n is negative, that means clockwise). One can see this directly, $e^{int} = \cos(nt) + i\sin(nt)$; or one can say that e^{it} goes once around the circle, and then taking the n-th power has the effect of multiplying the angles by n.

(14b) Roots and multiplicities. Take a complex polynomial of degree n:

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad \text{with} a_n \neq 0. \tag{14.5}$$

The fundamental theorem of algebra says that we can always write this as

$$f(z) = a(z - w_1)^{m_1} \cdots (z - w_k)^{m_k}, \tag{14.6}$$

where $a = a_n$ is the leading coefficient; the roots w_i are all different from each other; and the multiplicities m_i are positive integers, whose sum is n. This is a theoretical existence statement, which basically says that f has n zeros once those are counted with the proper multiplicities. If we know w is a root, we can actually compute its multiplicity without writing the polynomial in the form described above:

Lemma 14.1. *The multiplicity of a root w is the smallest m such that the m-th derivative of f at w is nonzero.*

EXAMPLE 14.2. *Take* $f(z) = z + 3z^2 - 3z^3 + z^4$*, which satisfies* $f(1) = 0$*. We compute*

$$\begin{aligned} f'(z) &= -1 + 6z - 9z^2 + 4z^3, & f'(1) &= 0, \\ f''(z) &= 6 - 18z + 12z^2, & f''(1) &= 0, \\ f'''(z) &= -18 + 24z, & f'''(1) &= 6 \neq 0, \end{aligned} \tag{14.7}$$

so the multiplicity at 1 *is* 3*.*

From now on, we write $\mathrm{mult}(f, w)$ for the multiplicity of f at w (if w is not a root, one can set that multiplicity to 0).

(14c) The winding number formula. Applying the same idea as in the previous lecture, we look at the image of a circle of radius $r > 0$ under f, which is the loop (with $T = 2\pi$)

$$d(t) = f(re^{it}) = f(r\cos(t) + ri\sin(t)). \tag{14.8}$$

Suppose that f has is no root on the circle of radius r around the origin, so that the winding number $\mathrm{wind}(d, 0)$ is defined. Then:

THEOREM 14.3. *For a loop* (14.8)*,*

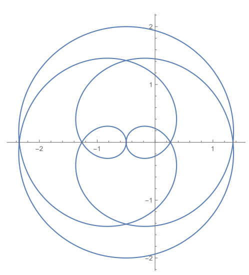
$$\mathrm{wind}(d, 0) = \sum_{\substack{|w|<r \\ f(w)=0}} \mathrm{mult}(f, w), \tag{14.9}$$

where the sum is over all roots of f *lying inside the circle of radius* r*.*

In particular, the winding number is always nonnegative; and it is > 0 *if and only if* there is a solution of $f(w) = 0$ inside the circle. This two-way implication is part of the special magic of the class of holomorphic functions, of which polynomials are the simplest examples.

EXAMPLE 14.4. *Take* $f(z) = z^5 - z^3 - \frac{1}{2}$*. The loop* $d(t) = f(e^{it}) = e^{5it} - e^{3it} - \frac{1}{2}$ *looks like this:*

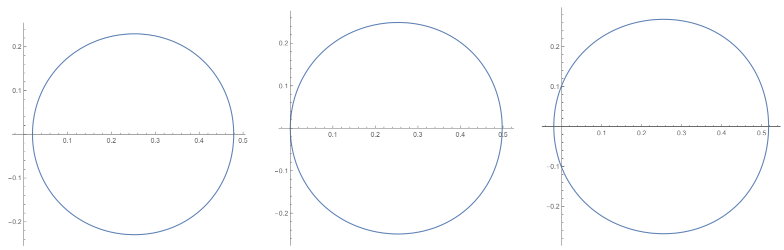
(14.10)



From that, one reads off the winding number around the origin, $\mathrm{wind}(d, 0) = 3$ *(the picture doesn't tell you which way the loop goes; but the other direction gives a winding number of* -3*, which is impossible). This means that we have three possibilities: either there are three solutions of* $f(p) = 0$ *with* $|p| < 1$*, each having multiplicity* 1*; or two solutions, with multiplicities* $1, 2$*; or a single solution, with multiplicity* 3 *(in fact, the first is the case, but you can't tell that just from our computation).*

EXAMPLE 14.5. *Take* $f(z) = (z + i)(z - i)(z + 1)(z - 1)(z - 1/4)$*. There is one root with* $|p| = 1/4$*, and four roots with* $|p| = 1$*. All roots have multiplicity* 1*. Consequently, the winding* *number* wind$(d, 0)$ *remains zero for* $r < 1/4$, *and then jumps to* 1. *The jump happens in a relatively simple way, by* d *moving across the origin:*

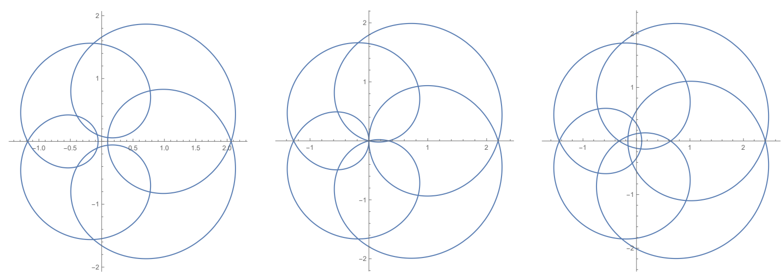
(14.11)



$r = 0.23$ $\qquad$ $r = 0.25$ $\qquad$ $r = 0.27$

The winding number remains at that value for $1/4 < r < 1$, *and then jumps to* 5 *when crossing* $r = 1$. *At that value, four parts of the loop* d *all pass through the origin simultaneously:*

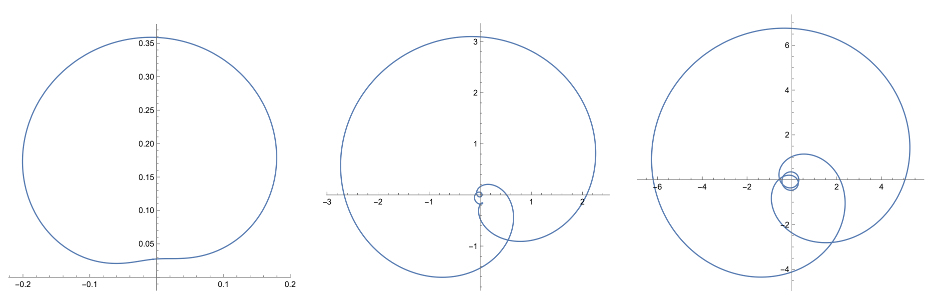
(14.12)



$r = 0.98$ $\qquad$ $r = 1$ $\qquad$ $r = 1.03$

Example 14.6. *Take* $f(z) = (z - \frac{1}{2})^3(z - i)$. *This has a root of multiplicity* 3 *at* $1/2$, *and a root of multiplicity* 1 *at* i. *Correspondingly, we expect the winding number to be* 0 *for* $r < 1/2$, *then* 3 *for* $r \in (1/2, 1)$, *and finally* 4 *for* $r > 1$. *The jump from* 0 *to* 3 *comes with a sudden curling behaviour:*

(14.13)



$r = 0.2$ $\qquad$ $r = 0.8$ $\qquad$ $r = 1.1$

(14d) Other values. We have focused on the equation $p(z) = 0$, but simply by subtracting a constant from p, one can apply the result to equations $p(z) = u$ for an arbitrary complex number u.

Corollary 14.7. *Take* $d(t) = p(re^{it})$ *as before. For every* u *where it is defined, the winding number* wind(p, u) *is nonnegative; and it is* > 0 *if and only if there is a solution of* $p(z) = u$ *with* z *inside the circle of radius* r.

EXAMPLE 14.8. *Take the leftmost picture from Example 14.5, with $r = 0.98$. We have not drawn that, but the motion of the loop is anticlockwise (meaning, to the left at its topmost point). As a consequence, one can check that the winding numbers are positive for all regions except the outermost infinite region. It follows that for any u lying in those regions, the equation $p(z) = u$ has a solution with $|z| < 0.98$.*

COROLLARY 14.9. *Take $d_1 = p(r_1 e^{it})$, $d_2 = p(r_2 e^{it})$, for some $r_2 > r_1 > 0$. Then, for every complex number u where both winding numbers are defined, we have*

$$\mathrm{wind}(d_2, u) \geq \mathrm{wind}(d_1, u). \tag{14.14}$$

In other words, as the radius increases, the winding number of our loops around any point can only go up or stay the same; it can never go down.

(14e) Proof. The proof of the theorem is, as usual for loops, by a deformation argument. We write our polynomial as a product, but separating the roots that lie inside and outside the circle of radius r:

$$f(z) = \Big(\prod_{|w_i|<r} (z - w_i)^{m_i}\Big)\Big(\prod_{|w_i|>r}\big)(z - w_i)^{m_i}. \tag{14.15}$$

Now we introduce a parameter $s \in [0, 1]$ which changes it like this:

$$f_s(z) = \Big(\prod_{|w_i|<r} (z - sw_i)^{m_i}\Big)\Big(\prod_{|w_i|>r} (sz - w_i)\Big). \tag{14.16}$$

If $|w_i| < r$ is a root of f, then $s|w_i|$ is a root of f_s. In the second instance, if $|w_i| > r$ is a root of f, then $|w_i|/s$ is a root of f_s (or for $s = 0$, there is no corresponding root). In words, the roots lying inside the circle of radius r move inwards as s becomes smaller, and those lying outside the circle move to infinity. At no time s does f_s actually have a root on the circle. Therefore, if we define

$$d_s(t) = f_s(re^{it}), \tag{14.17}$$

then $\mathrm{wind}(d_s, 0)$ remains the same for all s. For $s = 1$ we have $f_1 = f$, so $d_1 = d$ is the loop associated to the original polynomial. Now let's see what we get for $s = 0$:

$$f_0(z) = \Big(\prod_{|w_i|<r} z^{m_i}\Big)\Big(\prod_{|w_i|>r} (-w_i)^{m_i}\Big) = z^m a, \tag{14.18}$$

where $m = \sum_{|w_i|<r} m_i$, and $a = \prod_{|w_i|>r} (-w_i)^{m_i}$ is a nonzero constant. The associated loop is $d_0(t) = e^{imt}a$, which goes m times around the circle of radius $|c|$. Therefore, $\mathrm{wind}(d_0, 0) = m$. The same is therefore true of $\mathrm{wind}(d, 0) = \mathrm{wind}(d_1, 0)$. Looking at the definition of m, that's exactly what the theorem says!