15. The linking number

Mostly, we resist the urge to go off into higher dimensions; this lecture is one of the exceptions.

- Linking numbers are three-dimensional analogues of winding numbers, but the standard way of computing linking numbers involves projecting down to two dimensions.
- One way of defining the linking number is by an explicit integral. We use that definition to prove deformation invariance, and also to explain (partially) the projection formula.

(15a) Undercrossings and overcrossings. Take two loops in space,

$$\begin{aligned} c(t) &\in \mathbb{R}^3, \quad c(t+T) = c(t), \\ d(u) &\in \mathbb{R}^3, \quad d(u+U) = d(u), \end{aligned} \tag{15.1}$$

which never intersect each other:

$$c(t) \neq d(u) \quad \text{for all } t, u. \tag{15.2}$$

The linking number is an integer $\mathrm{link}(c,d) \in \mathbb{Z}$, which arises from trying pull the loops apart (without letting them intersect). Here are its basic properties:

- (Symmetry) $\mathrm{link}(c,d) = \mathrm{link}(d,c)$.
- (Direction) If we change the direction of one of the loops, meaning either replacing $c(t)$ with $c(-t)$, or $d(t)$ with $d(-t)$, the linking number switches sign (if we make both changes at the same time, the linking number remains the same).
- (Deformation invariance) The linking number remains constant under deformations of c and d, as long as the loops don't cross each other.
- (Separation) If c and d are separated by a plane (one loop on each side), the linking number is zero.

To compute the linking number, we project our loops to the (x,y) plane. This is as if we were looking at them from above (and from a great height). We'll need to impose two important requirements.

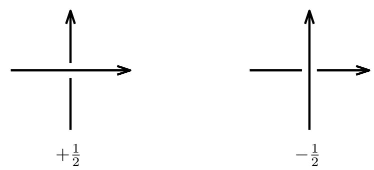
- Whenever the projected loops in the plane intersect each other, only one piece of c and one piece of d cross there. This means that if $c(t)$ and $d(u)$ have the same (x,y)-coordinates, then no other point of c or d has those (x,y)-coordinates.
- At any such a crossing point, the projected loops must cross each other transversally; meaning that their derivatives at the crossing point must be linearly independent.

One can deform (slightly) any given loops to achieve this. At each crossing point, we remember which loop was originally above the other in the z-coordinate. Then,

$$\mathrm{link}(c,d) = \sum_{\substack{\text{crossing points} \\ \text{of the projections}}} \pm\tfrac{1}{2}, \tag{15.3}$$

where the sign is determined by what the crossing looks like (including the directions of parametrization of the two loops):

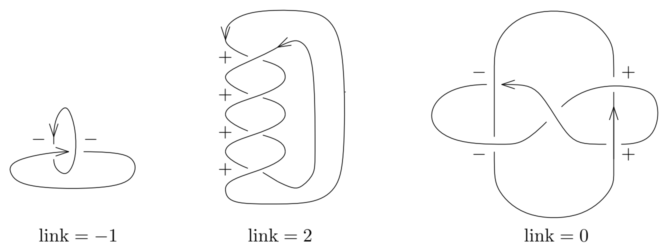
(15.4)



It doesn't matter which of the two pieces is c and which is d (remember the symmetry property), but it does matter that they belong to different loops: otherwise, we would have a selfintersection of one of our two projected loops, and *selfintersections do not count for the linking number.* One can show that there is always an even number of crossings (the polygonal analogue was Problem 4.7). This explains why, in spite of the $\frac{1}{2}$ in (15.3), the linking number is an integer.

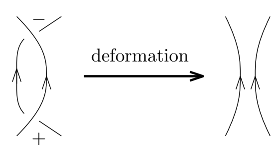
Example 15.1. *Here are some linking number computations:*

(15.5)



One can motivate the signs associated to the crossings by deformation invariance:

(15.6)



The undercrossing-overcrossing formula works well for computing the linking number, but is not a satisfactory theoretical basis, because it depends on a choice of projection to the plane.

(15b) The Gauss integral. Because we are working in three dimensions, the notational conventions are different from our usual ones. The length and dot (scalar) product are the obvious ones in $\mathbb{R}^3$, but $\times$ is now the spatial cross product

$$v \times w \in \mathbb{R}^3 \quad \text{for } v, w \in \mathbb{R}^3. \tag{15.7}$$

It is helpful to remember the following formula, which demystifies the cross product:

$$(v \times w) \cdot q = (\text{the determinant of the } 3 \times 3 \text{ matrix with column vectors } v, w, q). \tag{15.8}$$

One defines the linking number

$$\operatorname{link}(c, d) = \frac{1}{4\pi} \int_0^T \int_0^U \frac{(c'(t) \times d'(u)) \cdot (c(t) - d(u))}{\|c(t) - d(u)\|^3}\, du\, dt. \tag{15.9}$$

A priori this formula is mysterious, and it's not clear at all what it has to do with our previous description of the linking number. The first step in understand it to show that it has the required deformation invariance property:

THEOREM 15.2. *The integral* (15.9) *is unchanged if we deform* c *and* d*, as long as they don't cross each other.*

This can be proved by a lengthy multivariable computation, which we omit here. What's easier is the following consequence, which is a form of the separation property:

COROLLARY 15.3. *Suppose that* c *lies in* $\{z > 0\} \subset \mathbb{R}^3$*, and* d *in the region* $\{z < 0\} \subset \mathbb{R}^3$*. Then their linking number, as defined by* (15.9)*, is zero.*

PROOF. Let's use the deformation $c_s(t) = c(t) + (0,0,s)$ and $d_s(u) = d(u) - (0,0,s)$, which pulls the first loop up and pushes the second one down. This does not change the derivatives: $c_s'(t) = c'(t)$ and $d_s'(u) = d'(u)$. However, $\|c_s(t) - d_s(u)\|$ goes to infinity as s goes to infinity, because the two loops are separated by at least $2s$ distance. As a consequence, we have

$$\frac{c_s(t) - d_s(u)}{\|c_s(t) - d_s(u)\|^3} \longrightarrow 0 \quad \text{as } s \text{ goes to infinity.} \tag{15.10}$$

But that means that the entire integral goes to zero. On the other hand, by deformation invariance it doesn't change at all, which means it must have been equal to zero in the first place! □

(15c) Contributions from crossings. How does the definition via the integral formula relate to the original description of the linking number in terms of crossings? We will only look at this in a highly simplified toy model. Namely, let's suppose that instead of loops we consider the straight lines

$$c(t) = (t, 0, 0), \quad d(u) = (0, u, z), \tag{15.11}$$

where z is a nonzero constant. Then

$$\begin{aligned} &(c'(t) \times d'(u)) \cdot (c(t) - d(u)) = \det \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & -u \\ 0 & 0 & -z \end{pmatrix} = -z, \\ &\|c(t) - d(u)\| = \sqrt{t^2 + u^2 + z^2}. \end{aligned} \tag{15.12}$$

Of course, the linking number is not really defined in this situation; but we can still look at the integral, which is now an improper one, integrating over $(t,u) \in \mathbb{R}^2$. It can be explicitly solved by passing to polar coordinates:

$$\begin{aligned} -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z}{(z^2 + t^2 + u^2)^{3/2}}\, dt\, du &= -\tfrac{1}{2} \int_0^{\infty} \frac{zr}{(z^2 + r^2)^{3/2}}\, dr \\ &= \tfrac{1}{2} \frac{z}{(z^2 + r^2)^{1/2}} \Big|_{r=0}^{r=\infty} = -\tfrac{1}{2} \frac{z}{|z|} = \begin{cases} -\frac{1}{2} & z > 0, \\ \frac{1}{2} & z < 0. \end{cases} \end{aligned} \tag{15.13}$$

These are exactly the contributions from (15.4), for the unique crossing of our paths-which-are-not-loops.

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