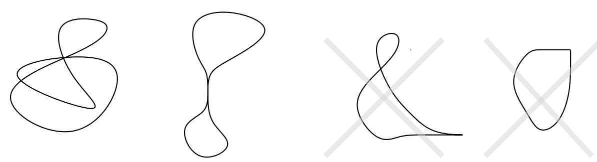
## 16. Immersed loops and the rotation number

Possibly, our notion of loop will have struck you as intuitively wrong, since it allows for example a constant map to be a loop. That may be because what you have in mind is a different notion, that of an immersed loop. Briefly, an immersed loop is one that can't stop moving at any time.

- Immersed loops have another topological invariant associated to them, the rotation number, which involves the direction of motion rather than the position.
- The rotation number can be easily explained in terms of the winding number of derivatives, and then we can use our previous techniques to compute it.
- There is another and more exciting way of computing the rotation number, in terms of selfintersection points.

**(16a) Immersed loops.** Take a loop $c$ with period $T$, $c(t+T) = c(t) \in \mathbb{R}^2$. We say that the loop is *immersed* if $c'(t) \in \mathbb{R}^2$ never becomes zero. An immersed loop can have selfintersection or self-tangency points. It can even repeat the same trajectory several times. What it can't do is form a kind of corner: it always moves forward in the direction of its tangent line, which itself varies differentiably in time. Here are some examples and non-examples of immersed loops:

(16.1)



It's important to remember that the two non-examples on the right could in principle be smooth loops, parametrized in such a way that the derivative is zero at the kinks in the image.

We define the *rotation number* of an immersed loop by the integral formula

$$\mathrm{rot}(c) = \frac{1}{2\pi}\int_0^T \frac{c'(t)\times c''(t)}{\|c'(t)\|^2}\,dt. \tag{16.2}$$

Example 16.1. *Take*

$$c(t) = (R\cos(mt), R\sin(mt)) \quad (T = 2\pi), \tag{16.3}$$

*the loop that goes $m$ times round a circle of radius $R$. Then $\|c'(t)\| = Rm$, and*

$$c'(t)\times c''(t) = \begin{pmatrix} -Rm\sin(mt) \\ Rm\cos(mt)\end{pmatrix} \times \begin{pmatrix} -Rm^2\cos(mt) \\ -Rm^2\sin(mt)\end{pmatrix} = R^2m^3, \tag{16.4}$$

*so*

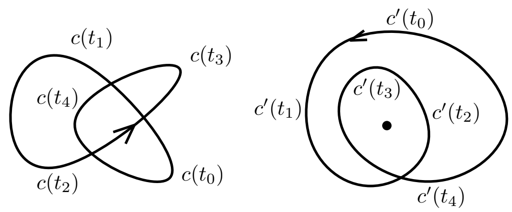
$$\frac{1}{2\pi}\int_0^T \frac{c'(t)\times c''(t)}{\|c'(t)\|^2}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2m^3}{(Rm)^2}\,dt = m. \tag{16.5}$$

**(16b) Counting tangencies.** Saying that a loop is immersed means exactly that $c'(t) \in \mathbb{R}^2$ avoids the origin $o$. It is therefore natural to suspect a relation with the winding number of $c'$. Indeed, a direct comparison of the integral formulae shows that

$$\mathrm{rot}(c) = \mathrm{wind}(c', o). \tag{16.6}$$

Example 16.2. *Here's a trefoil-shaped immersed loop* $c(t)$*, and an approximate picture of* $c'(t)$*:*

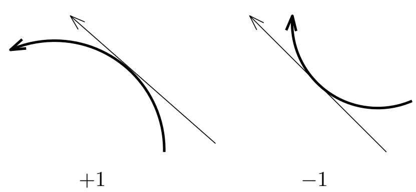
(16.7)




*From the right-hand picture, we read off that* $\mathrm{rot}(c) = \mathrm{wind}(c', o) = 2$*.*

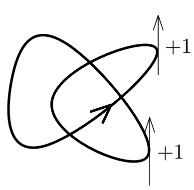
Recall that, to compute the winding number of a loop $d(t)$ around a point, we send out a ray from that point to infinity and, roughly speaking, count the intersections of the ray with the loop. If our point is the origin, and the ray goes in direction $w$, this amounts to looking at all $d(t)$ which are positive multiples of $w$. Applying the previous recipe to $d(t) = c'(t)$, we get a way of computing the rotation number of $c$ by counting points where $c'(t)$ points in some chosen direction. Let's explain the outcome in a self-contained way. Let $c$ be an immersed loop. Choose a nonzero vector $w$, and look at those $t$ where $c'(t)$ points in the same direction as (in other words, is a positive multiple of) $w$. We count those with signs:

(16.8)




Example 16.3. *Here's another way of computing that the trefoil loop has rotation number* 2*, by counting upwards-pointing tangencies:*

(16.9)




There is an implicit assumption here, which is that $w$ is chosen so that our curve bends either to the left or to the right at the relevant points. Let's make this a little more rigorous. The assumption is:

(16.10) for each $t$ such that $c'(t)$ is a positive multiple of $w$, the vectors $(c'(t), c''(t))$ must be linearly independent.

Then, we can write our formula as

$$\mathrm{rot}(c) = \sum_t \mathrm{sign}(c'(t) \times c''(t)), \tag{16.11}$$

summing over all $t \in [0, T)$ which appear in (16.10).

**(16c) Counting selfintersections.** Take an immersed loop $c$, with period $T$. We say that the loop is *embedded* if it has no selfintersections. This means that for each $q \in \mathbb{R}^2$, there is *at most* one $t \in [0, T)$ such that $c(t) = q$. One can think of embedded loops as the smooth analogues of polygons.

Proposition 16.4. *(Umlaufsatz) For an embedded loop $c$, one always has* $\mathrm{rot}(c) = \pm 1$.

We get $+1$ if we go around the loop anticlockwise, $-1$ if we go around the loop clockwise. Even though that may not be clear at first sight, this is the curved analogue of the familiar fact that the angles of a polygon with $n$ vertices add up to $(n-2)\pi$ (if you want to explore the connection, you'd have to approximate our loops by a polygonal one, and then see how the polygonal approximation to the rotation number is related to the angles).

There is a generalization of the Umlaufsatz, which yields a remarkable relation between the rotation number and selfintersection points. Let $c(t)$ be an immersed loop (with period $T$). We say that $c$ has *simple selfintersections* if the following two conditions hold:

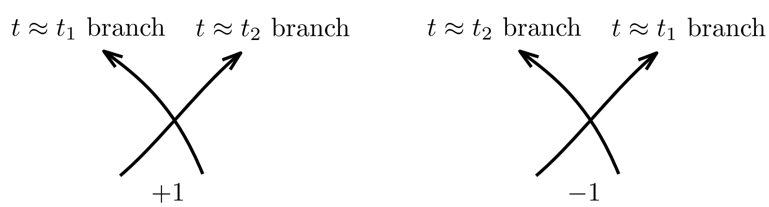
- (No triple intersections) For every $q \in \mathbb{R}^2$, there are at most two $t \in [0, T)$ with $c(t) = q$.
- (Transverse crossing) If $q = c(t_1) = c(t_2)$ for some $t_1 < t_2$ in $[0, T)$, then $(c'(t_1), c'(t_2))$ must be linearly independent.

Suppose that we have a selfintersection point, with notation as in the second condition above. We can give it a sign,

$$\sigma(q) = \mathrm{sign}(c'(t_2) \times c'(t_1)) = -\mathrm{sign}(c'(t_1) \times c'(t_2)) \in \{\pm 1\}. \tag{16.12}$$

Geometrically, the convention looks like this, remembering always that $t_1 < t_2$:

(16.13)




The sign is sensitive to where we put the starting point $t = 0$ on the loop. We want to choose that in a particular way:

(16.14) An immersed loop is said to have an outside starting point if the entire loop lies in the half-plane to one side of its tangent line at $t = 0$.

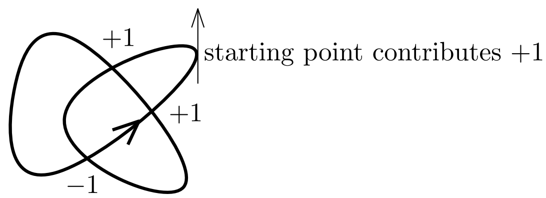
Theorem 16.5. *(Whitney's formula) Let $c$ be an immersed loop with simple selfintersections, and which has an outside starting point. Then,*

$$\mathrm{rot}(c) = \pm 1 + \sum_q \sigma(q). \tag{16.15}$$

*The sign of the first term is fixed as follows. Start at $c(0)$, and look in direction $c'(0)$. If the loop lies in the half-plane to the left of you, we get $+1$; if it lies in the half-plane to the right of you, we get $-1$. The other terms are the signs associated to selfintersection points.*

Example 16.6. *Yet again, we compute the rotation number of the trefoil loop, this time using Whitney's formula:*

(16.16)




$$+1+1-1+1=2.$$

**(16d) Deformations.** The simple selfintersection requirement may strike one as a problem: what if it fails? Here, deforming immersed loops comes in handy.

Fact 16.7. *The rotation number is deformation invariant within the class of immersed loops. This means that if $c_s(t)$ is a deformation of loops ($0 \leq s \leq 1$), such that for every value of the parameter $s$ the loop $t \mapsto c_s(t)$ is immersed, then $\mathrm{rot}(c_0) = \mathrm{rot}(c_1)$.*

This follows immediately from the corresponding property of the winding number. Saying that the loops $c_s$ must remain immersed is the same as saying that $c_s'(t) = \partial c_s(t)/\partial t$ must avoid the origin, which is what we need in order for $\mathrm{wind}(c_s', o)$ to be constant as a function of $s$.

Example 16.8. *The loop drawn below can be deformed to a circle without breaking immersion (to see that, imagine pushing each of the four pieces which stick out along the coordinate axes all the way through to the opposite side). Therefore, it has rotation number* 1.

(16.17)