17. Arnold invariants

We mentioned in the last lecture that deforming an immersed loop preserves its rotation number. Actually, the converse is also true: two loops with the same rotation number can be deformed into each other through immersed loops. One could think that this finishes off the topic of the topology of immersed loops, but actually, it's only the beginning!

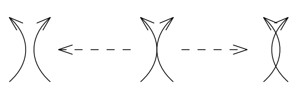
- Any deformation can be broken down into steps, each of which changes the picture of the loop in the plane in a specific way. There are three kinds of such transitions.
- There is a way of keeping track of how many moves of each kind occur. This is provided by Arnold invariants, one for each kind of transition.

(17a) Deformations of immersed loops. We begin by stating (without proof) the result mentioned above:

Theorem 17.1. *(Whitney-Graustein) Take two immersed loops $c_0(t)$ and $c_1(t)$ with (the same period T, and) the same rotation number. Then, one can deform one into the other through immersed loops.*

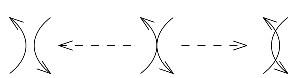
To get a more explicit picture, let's start with the notion of immersed loop with simple selfintersections, from the last lecture. What are the simplest ways in which an immersed loop can *fail* to have simple selfintersections? The loop could have a self-tangency, meaning two points $0 \leq t_1 < t_2 < T$ such that $c(t_1) = c(t_2)$ and the derivatives $c'(t_1)$, $c'(t_2)$ are linearly dependent. More precisely, this comes in two versions, the *direct self-tangency*, where $c'(t_1)$, $c'(t_2)$ differ by multiplication by a positive number; and the *inverse self-tangency*, where the number is negative. In both cases, we can perturb the loop a little to remove the self-tangency. Depending on how we perturb, we may or may not create a pair of selfintersection points:

(17.1)



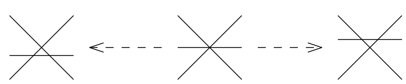
and

(17.2)



The other simple way in which our loop could fail to have simple selfintersections is by having a triple intersection point. Again, there are two ways of perturbing this situation to get rid of the triple point. Each of them trades it for a triple of simple selfintersection points:

(17.3)



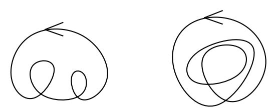
One can use this to give the following more combinatorial version of Whitney-Graustein:

Theorem 17.2. *Take two immersed loops with only simple selfintersections, and which have the same rotation number. Then, they can be transformed into each other by a composition of the following kinds of deformations:*

- *Deformations during which the loop retains its property of having simple selfintersections at all times. Such deformations do not change the overall topological picture of the loop in the plane. In particular, they can't create or destroy selfintersections. In the game we're about to play, this is considered a "free action" which you can do at any time.*
- Direct self-tangency moves, *which means passing from one side of* (17.1) *to the other. This creates or destroys a pair of selfintersection points.*
- Inverse self-tangency moves, *which are the same for* (17.2)
- Triple point moves, *which means passing from one side of* (17.3) *to the other. This preserves the number of selfintersection points.*

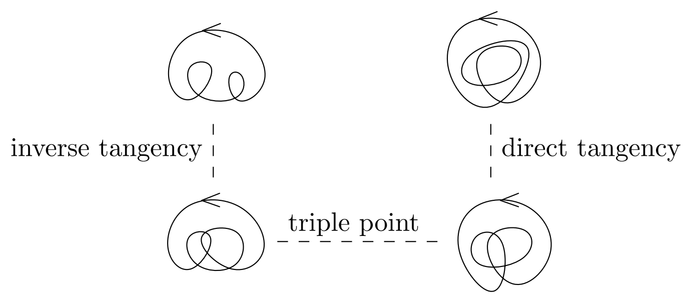
EXAMPLE 17.3. *These two loops have rotation number* 3*, and* 2 *simple selfintersection points:*

(17.4)



They are related by a sequence of three moves, one of each type:

(17.5)



Of course, this is only one way of constructing a deformation, there are many others. One can ask, for instance: can we transform those two loops into each other using only triple point moves?

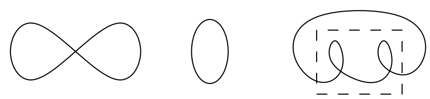
(17b) The $J^{\pm}$ invariants. We now introduce the first two Arnold invariants, which count self-tangency moves (with suitable signs).

PROPOSITION 17.4. *To each immersed loop c with simple selfintersections one can associate two integers $J^-(c)$ and $J^+(c)$, such that the following are satisfied:*

- *Reversing the direction of a loop doesn't change J^- or J^+.*
- *The loops below have prescribed values of the invariants:*

$J^- = -1$, $J^+ = 0$ $\quad$ $J^- = 0$, $J^+ = 0$ $\quad$ $J^- = -3k$, $J^+ = -2k$

(17.6)



k curls (as drawn, $k = 2$)

- *Under an inverse self-tangency move which creates two new selfintersection points, J^- decreases by* 2 *(conversely, under such a move which destroys two selfintersection points, it increases by* 2*); and J^+ remains the same.*

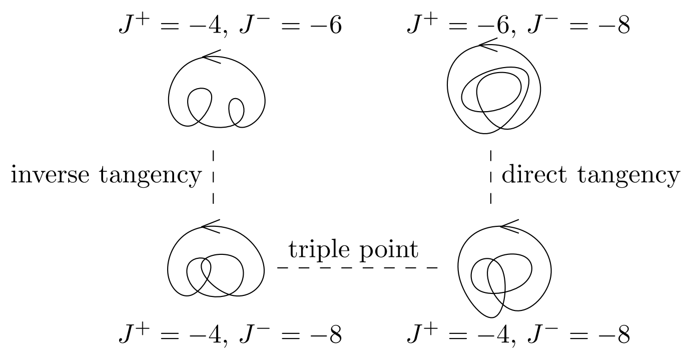
- *Under a direct self-tangency move which creates two new selfintersection points, J^+ increases by 2 (conversely, under such a move which destroys two selfintersection points, it decreases by 2); and J^- remains the same.*
- *J^- and J^+ do not change under triple point moves.*

The first property gives one example for each winding number. Together with the rules for what happens under the different types of moves, this describes J^- and J^+ completely, and allows us to compure it in any given case, by finding a deformation that transforms the loop into the relevant example where the values are given. In practice, you only have to know one of the two invariants, because (as one sees by looking at the moves):

FACT 17.5. *$J^+(c) - J^-(c)$ is the number of selfintersection points of c.*

EXAMPLE 17.6. *Let's revisit our previous example. The $J^\pm$ values for one loop are prescribed by* (17.6)*, and we derive those for the other loop by following the moves:*

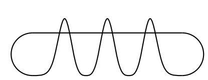
(17.7)



Hence, it's impossible to relate the two loops without using self-tangency moves of both kinds!

EXAMPLE 17.7. *Take this loop:*

(17.8)

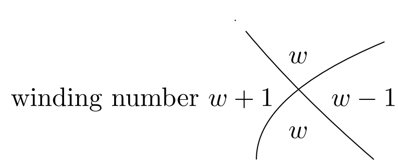


$2k$ selfintersections (here, $k = 3$)

Obviously, one can deform this to a circle (an embedded loop) by using k inverse self-tangency moves, and no moves of any other kind. Hence, $J^+ = 0$, just by looking up the value for the circle. Moreover, $J^- = -2k$, which one can either see by following the moves, or from Fact 17.5.

(17c) An explicit formula. If we have a simple selfintersection point q, the winding numbers for the regions surrounding q form a pattern like this:

(17.9)



(we saw this first in the corresponding polygonal case). Let's call w the mean winding number of c at q, written as meanwind(c, q).

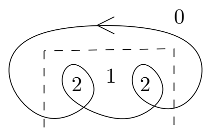
Proposition 17.8. *(Viro-Gutkin) One can compute the J^--invariant by the formula*

$$J^-(c) = 1 - \sum_R \mathrm{wind}(c,R)^2 + \sum_q \mathrm{meanwind}(c,q)^2. \tag{17.10}$$

Here, R are the regions into which c divides the plane; we take a point in each region, and $\mathrm{wind}(c,R)$ *is the winding number around that point. The second sum is over selfintersections q.*

Example 17.9. *Let's see that this is compatible with the values originally stated. The loop*

(17.11)

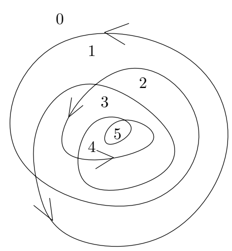


with k curls divides the plane into $k+2$ regions with winding numbers $0, 1$ and 2 (the latter appears k times). The k selfintersection points all have mean winding number 1. Therefore

$$J^- = 1 - (1 + 4k) + k = -3k. \tag{17.12}$$

Example 17.10. *Take the following (with rotation number 5):*

(17.13)



The regions of its complement have winding numbers $0, 1, 2, 3, 4, 5$, as drawn in the picture; and the selfintersection points have mean winding numbers $1, 2, 3, 4$. As a consequence

$$J^- = 1 - (1^2 + 2^2 + 3^2 + 4^2 + 5^2) + (1^2 + 2^2 + 3^2 + 4^2) = -24, \tag{17.14}$$

and therefore $J^+ = -20$. It follows that, in order to deform it to the "standard loop" which is the $k = 4$ case of (17.6) with $J^- = -12$ and $J^+ = -8$, we need at least 6 direct and 6 inverse self-tangency moves.

18.900 Geometry and Topology in the Plane
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