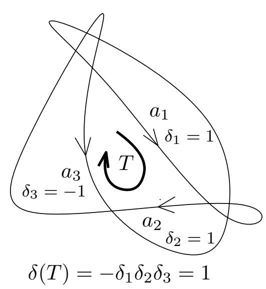
18. Arnold invariants (continued)

Among the moves introduced in the previous lecture, the triple point move is the most mysterious one: self-tangency moves create, or in reverse direction destroy, self-intersection points, but at first sight, the triple point move doesn't have such a directionality.

- Nevertheless, a closer look at the situation allows us to assign a sign to such a move. This underlies the third Arnold invariant, called *strange invariant*, which was missing from our collection so far.
- There is an explicit expression for this invariant, which combines the signs from Whitney's formula for the rotation number with the (mean) winding number.

(18a) Triangles. Let c be an immersed loop with simple self-intersections. Suppose that among the regions into which it divides the plane, there is a triangular one T, where the vertices of the triangle correspond to three different selfintersection points.

(18.1)



As we go around the loop, we will pass through the three sides of the triangle in some order: let's number the sides correspondingly as a_1, a_2, a_3 (if we move the starting point of c elsewhere, we might get to the sides in order (a_2, a_3, a_1) or (a_3, a_1, a_2) instead, but that turns out not to matter in the end). Let's just focus on the triangle, and go around it in the same order (a_1, a_2, a_3). We set $\delta_1 = 1$ if that way of going around the triangle agrees with the direction of a_1 (increasing t), and $\delta_1 = -1$ otherwise. Similarly, we have δ_2 and δ_3. We define the sign of the triangle to be

$$\delta(T) = -\delta_1\delta_2\delta_3 \in \{\pm 1\}. \tag{18.2}$$

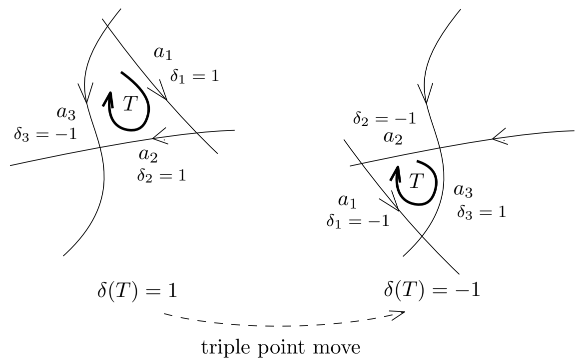
FACT 18.1. *The sign of the triangle does not change if we reverse the direction of the loop, meaning that we replace $c(t)$ with $c(-t)$.*

Passing to $c(-t)$ means that we reach the sides of T in the opposite order; but at the same time, the increasing t-direction on each side reverses; so the δ_k remain the same.

FACT 18.2. *Suppose that we carry out a triple point move with our triangle. After that move, we get a new triangle whose sign is the opposite of the old one.*

This is not hard to see. All of the δ_k change signs after the move, with the compound effect being a sign change in $\delta(T)$:

(18.3)

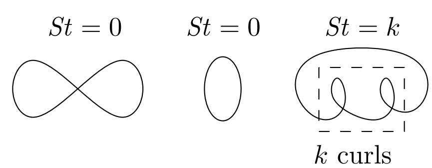


(18b) The strange invariant. The observation above provides triple point moves with a kind of directionality: if one such move replaces a negative triangle with a positive one, then the reverse move does the opposite.

THEOREM 18.3. *To every immersed loop c with simple selfintersections, one can associate an integer $St(c)$, such that the following are satisfied:*

- *Reversing the direction of a loop doesn't change St.*
- *The loops below have prescribed St invariants:*

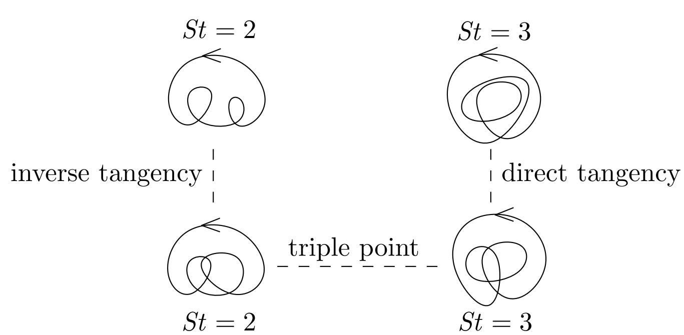
(18.4)



- *Under a triple point move which replaces a negative triangle with a positive one, St increases by 1 (and the reverse direction decreases it by 1).*
- *St does not change under the other kinds of moves.*

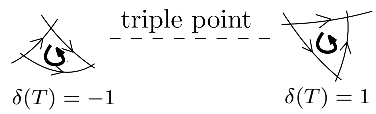
EXAMPLE 18.4. *We look at the same series of moves as last lecture, starting with one of the standard examples. The triple point move changes a triangle with $\delta(T) = 1$ into one with $\delta(T) = -1$, hence decreases the strange invariant by 1.*

(18.5)



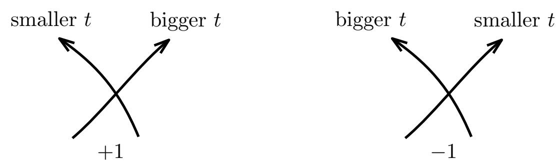
To see that St indeed goes up (and now down) by 1, we need to look at the signs of the triangles that are involved:

(18.6)



(18c) An explicit formula. Suppose that c has an outside starting point. Then, every selfintersection point has a sign $\sigma(q) \in \{\pm 1\}$, which is what appears in Whitney's formula for the rotation number. As a reminder, the conventions are:

(18.7)



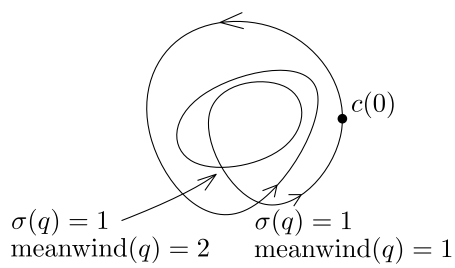
We also have the mean winding number for selfintersection points, introduced in the previous lecture. The two combine in this formula:

Proposition 18.5. *(Shumakovich) Assuming an exterior starting point, one can compute the strange invariant by*

$$St(c) = \sum_q \sigma(q)\,\mathrm{meanwind}(c, q). \tag{18.8}$$

Example 18.6. *We reconsider the previous computation:*

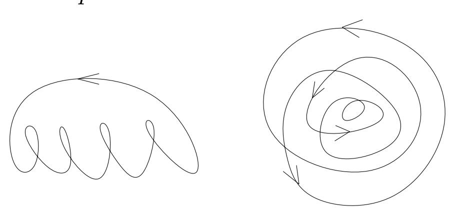
(18.9)



There are two double points, whose contributions yield $St(c) = 1 + 2 = 3$.

Example 18.7. *Take these two loops:*

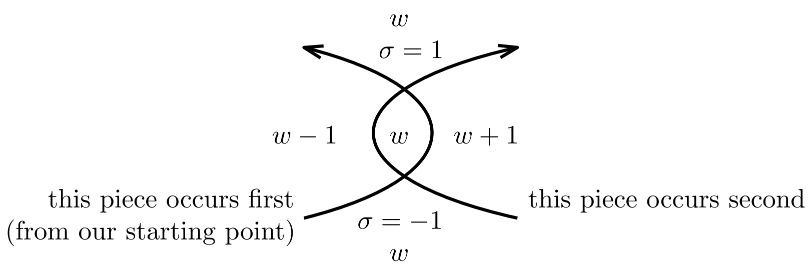
(18.10)



They both have 4 selfintersection points, and rotation number 5, so the Whitney signs are clearly all $\sigma(q) = 1$. On the left, the mean indices are all 1, yielding $St = 4$. On the right, the mean indices are $1, 2, 3, 4$, so $St = 10$. This means that when transforming one loop into the other, at least 6 triple point moves are necessary. We proved in the last lecture that at least 6 direct and 6 inverse self-tangencies are necessary, so in total, one needs at least 18 moves (this is a lower bound; I haven't checked whether you can really get away with just that many moves).

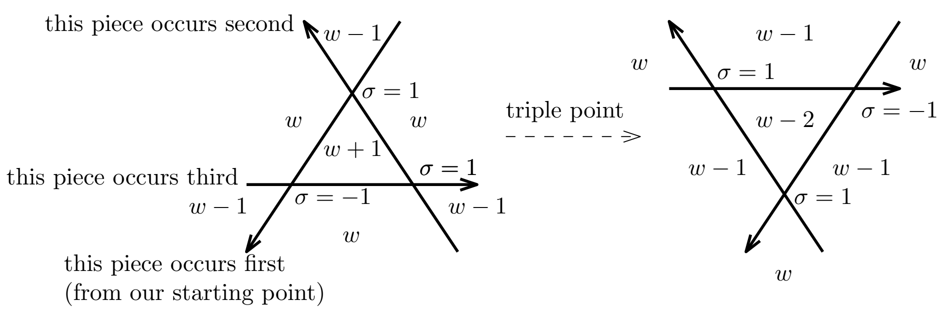
To see why Proposition 18.5 is correct, one needs to look at the behaviour of the proposed formula under the various moves. A direct or inverse tangency creates two new selfintersection points, which have opposite signs $\sigma(q)$ (after all, the rotation number doesn't change, so their contributions to Whitney's formula must cancel out) and the same meanwind(q). Therefore, the right hand side of (18.8) is invariant under self-tangency moves, as it should be. Here's a sample picture of the situation (the w are winding numbers of the regions, and the σ the Whitney signs of the selfintersection points):

(18.11)



Here's an example which shows how to analyze the effect of a triple point move:

(18.12)



The Whitney signs carry over, but the winding numbers change. In this particular case, the mean winding numbers of all three intersection points go down by 1, but since the Whitney signs are $(+1, +1, -1)$, the cumulative effect is that the right hand side of (18.8) decreases by 1; which, as one can check, matches the sign convention for St.