19. Introduction to algebraic curves

In this chapter we will discuss algebraic curves in the plane, which are described by a polynomial equation in two variables. To familiarize you with this kind of object, this lecture is dedicated to fairly basic example constructions:

- We look at algebraic curves which are of the very special form $f(x) = g(y)$, and how one can draw roughly what such a curve looks like.
- We show one can construct algebraic curves passing through a given finite collection of points in the plane (interpolation).
- We look at other ways in which algebraic curves can arise, through rational or trigonometric parametrizations.

(19a) Definition and first examples. An algebraic curve is the subset of the plane formed by the solutions (x, y) of a non-constant polynomial equation in two variables,

$$C = \{(x, y) \in \mathbb{R}^2 \;:\; f(x, y) = 0\}. \tag{19.1}$$

To make our language more precise, let's say that $x^i y^j$ is a monomial of degree $i + j$. Then, a polynomial of degree $\leq d$ is a sum of monomials

$$f(x, y) = \sum_{\substack{i \geq 0, j \geq 0 \\ i+j \leq d}} a_{ij} x^i y^j, \tag{19.2}$$

with real coefficients $a_{ij} \in \mathbb{R}$. If at least one of the top degree terms $a_{ij} x^i y^j$, $i + j = d$, is nonzero, we say that the polynomial has degree d. This notion of degree behaves in the way familiar from a single variable: if you multiply polynomials, the degrees add. An *algebraic curve of degree* $d > 0$ is the zero-set of a polynomial of that degree. For instance, a degree 1 polynomial is just a linear function $f(x, y) = a_{00} + a_{10}x + a_{01}y$, with $(a_{10}, a_{01}) \neq (0, 0)$; and hence, a degree 1 algebraic curve is just a straight line. We call a degree 2 algebraic curve a conic.

FACT 19.1. *The conics are of the following kinds:*

- *ellipses, including circles;*
- *parabolae;*
- *hyperbolae; these three cases together are the* classical conics.
- *Unions of two lines. This happens when $f(x, y)$ is the product of two degree 1 polynomials. The two lines can intersect (for instance, $xy = 0$), or they can be parallel ($x(x - 1) = 0$), or they can even be the same ($x^2 = 0$; in situations like this, the terminology "curve of degree d" becomes a little awkward).*
- *Sets consisting of one point in the plane ($x^2 + y^2 = 0$).*
- *The empty set ($x^2 + y^2 = -1$).*

As we saw above, the union of two lines is an algebraic curve. More generally,

FACT 19.2. *If C_1 and C_2 are algebraic curves, then so is $C = C_1 \cup C_2$. To see that, write $C_i = \{f_i(x, y) = 0\}$, and then $C = \{f(x, y) = 0\}$, where $f(x, y) = f_1(x, y) f_2(x, y)$.*

We also saw that a single point is an algebraic curve. This is also an instance of a wider observation:

FACT 19.3. *If C_1 and C_2 are algebraic curves, then so is $C = C_1 \cap C_2$. To see that, write $C_i = \{f_i(x,y) = 0\}$, and then $C = \{f(x,y) = 0\}$, where $f(x,y) = f_1(x,y)^2 + f_2(x,y)^2$.*

We've defined an algebraic curve just as a subset $C \subset \mathbb{R}^2$ which can be described by an algebraic equation, but different equations can give the same curve. The unfortunate outcome of this is that the degree of C is in general ambiguous. The line $\{x = 0\}$ is also the conic $\{x^2 = 0\}$, and indeed the degree n curve $\{x^n = 0\}$ for any n.

The following is maybe the simplest way to construct examples of higher degree curves whose structure you can understand. Look at

$$C = \{p(x) = y^2\}. \tag{19.3}$$

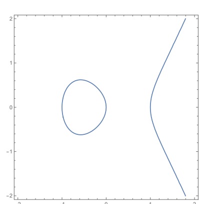
This means that $y = \pm\sqrt{p(x)}$, so for every x, we have $0, 1$ or 2 solutions of y, depending on the sign of $p(x)$.

EXAMPLE 19.4. *Take $C = \{x^3 - x = y^2\}$. This satisfies*

$$p(x) = x^3 - x = (x-1)(x+1)x \begin{cases} \textit{negative} & x < -1, \\ \textit{positive} & -1 < x < 0, \\ \textit{negative} & 0 < x < 1, \\ \textit{positive} & x > 1, \\ \textit{zero} & x = -1, 0, 1. \end{cases} \tag{19.4}$$

So, we get two solutions of $y^2 = p(x)$ for every $x \in (-1, 0)$, and also for every $x > 1$. Here's what C actually looks like:

(19.5)



(19b) Interpolation. Everyone knows that there's a line through any two given points. The result one degree higher is this:

LEMMA 19.5. *For any 5 points in the plane, there is a conic which goes through all those points. (There may be more than one, depending on the positions of the points, but there is at least one. It may not be a classical conic, though.)*

To prove the result, write $q_i = (x_i, y_i)$, $i = 1, \ldots, 5$. Let's look at a general conic $f(x,y) = 0$,

$$f(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}. \tag{19.6}$$

The condition for that conic to go through our five points are

$$
\begin{aligned}
&a_{20}x_1^2 + a_{11}x_1y_1 + a_{02}y_1^2 + a_{10}x_1 + a_{01}y_1 + a_{00} = 0,\\
&a_{20}x_2^2 + a_{11}x_2y_2 + a_{02}y_2^2 + a_{10}x_2 + a_{01}y_2 + a_{00} = 0,\\
&a_{20}x_3^2 + a_{11}x_3y_3 + a_{02}y_3^2 + a_{10}x_3 + a_{01}y_3 + a_{00} = 0,\\
&a_{20}x_4^2 + a_{11}x_4y_4 + a_{02}y_4^2 + a_{10}x_4 + a_{01}y_4 + a_{00} = 0,\\
&a_{20}x_5^2 + a_{11}x_5y_5 + a_{02}y_5^2 + a_{10}x_5 + a_{01}y_5 + a_{00} = 0.
\end{aligned} \tag{19.7}
$$

These are 5 linear equations for the 6 unknown coefficients of the conic. Hence, there must be a solution where not all of the a_{ij} are zero. (In principle, the resulting $f(x,y)$ could have degree 1, but then one could take the product with an arbitrary linear term to get the degree back up to 2). The same idea actually works in any degree:

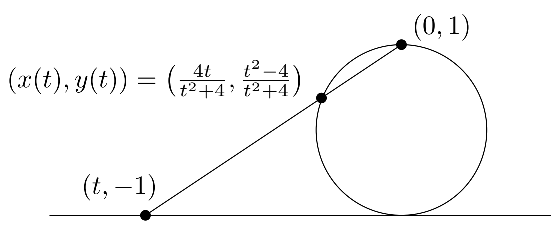
THEOREM 19.6. *Take some $d \geq 1$, and choose $d(d+3)/2$ points in the plane. Then there is an algebraic curve of degree d which passes through all of them.*

(19c) Parametrizations. We are used to two ways of describing curves, one by equations and the other by parametrizations. Algebraic curves are by definition given by polynomial equations, but we also have the following:

THEOREM 19.7. *Any two rational functions $x(t)$ and $y(t)$ parametrize part of an algebraic curve.*

EXAMPLE 19.8. *One can find a rational parametrization of the circle $x^2+y^2=1$ (minus a point) as follows. Draw a line from a point $(t,-1)$ to $(0,1)$. This line intersects the circle at one point other than $(0,1)$, and one can solve for the coordinates of that point:*

(19.8)



To understand why the theorem holds, let's suppose that $x(t)$ and $y(t)$ are polynomials of degree ≤ 3. We claim that then, they parametrize part of an algebraic curve of degree ≤ 4. Look at the monomials that can occur,

$$
\begin{gathered}
1,\ x(t),\ y(t),\ x(t)^2,\ x(t)y(t),\ y(t)^2,\ x(t)^3,\ x(t)^2y(t),\ x(t)y(t)^2,\ y(t)^3,\\
x(t)^4,\ x(t)^3y(t),\ x(t)^2y(t)^2,\ x(t)y(t)^3,\ y(t)^4.
\end{gathered} \tag{19.9}
$$

Each of these is a polynomial of degree $\leq 3 \cdot 4 = 12$ in t. Such a polynomial has 13 coefficients, so we can think of it as a vector in $\mathbb{R}^{13}$. There are 15 such polynomials/vectors, so there must be a linear relation between them $(15 > 13)$; and that translates into a polynomial relation between $x(t)$ and $y(t)$ of degree ≤ 4. The same argument shows that if $x(t)$ and $y(t)$ are polynomials of degree $\leq d$, for some $d \geq 2$, then they trace out part of an algebraic curve of degree $\leq 2d-2$. The general case of rational functions is similar but more complicated, since one has to take the degrees of numerator and denominator into account.

A trigonometric polynomial of degree $\leq d$ is an expression

$$p(\theta) = a + \sum_{k=1}^{d} b_k \cos(k\theta) + \sum_{k=1}^{d} c_k \sin(k\theta), \tag{19.10}$$

with constants $a, b_1, \ldots, b_d, c_1, \ldots, c_d$. A trigonometric rational function is then defined as a function that's a quotient of two trigonometric polynomials.

Theorem 19.9. *Any two trigonometric rational functions $x(\theta)$ and $y(\theta)$ parametrize part of an algebraic curve.*

One can prove this as before, by finding linear relations between the $x(\theta)^i y(\theta)^j$ (by the angle addition formulae, these are all polynomials in $\cos(\theta)$ and $\sin(\theta)$). There is also a more sneaky approach, using the rational parametrization $(x(t), y(t))$ of the circle from (19.8): substituting $(\cos(\theta), \sin(\theta)) = (x(t), y(t))$ turns a parametrization by trigonometric rational functions into one (much more complicated) by ordinary rational functions. So, the theorem can actually be reduced to the previous one.

Any kind of converse to the theorems above is false: "most" algebraic curves of degree > 2 can't be parametrized by rational (or trigonometric rational) functions. In other words, any parametrization of such a curve must be by functions which are more complicated.

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Spring 2023

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