2. Integer polygons

An integer polygon is one whose vertices (corners) have integer coordinates.

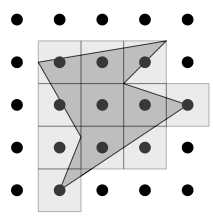
- We discuss Pick's theorem on the area of integer polygons;
- We consider what coordinate transformations one can apply to integer polygons, and look specifically at integer triangles.

(2a) Computing areas. Let's say we want to estimate the area of a polygon. Here's one way:

$$\text{area}(P) \approx (\text{number of integer points in } P). \tag{2.1}$$

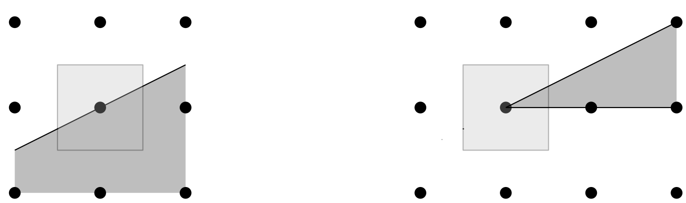
The idea is that around each integer point (point with integer coordinates) we draw a 1×1 square. The collection of those squares approximates the polygon:

(2.2)



Our formulation "in P" left it ambiguous what to do with integer points that lie on the boundary of P. We can improve the formula by treating them differently from the integer points in the interior. If an integer point lies on one of the edges (sides) of P, but is not a vertex (corner), let's count it as $\frac{1}{2}$ (instead of 1 for points in the interior). If an integer point is a vertex of P, with interior angle α, we count it as $\alpha/2\pi$. Here are two examples of how points contribute to our new formula:

(2.3)



contribution $1/2$ contribution $\arctan(1/2)/2\pi$

The formula is:

$$\begin{aligned} \text{area}(P) \approx\ & (\text{number of integer points in the interior of } P) \\ & + \tfrac{1}{2}(\text{number of integer points on the boundary of } P\text{, which are not vertices}) \\ & + \tfrac{1}{2\pi}(\text{sum of interior angles at all integer vertices}). \end{aligned} \tag{2.4}$$

Of course, this idea is still only an approximation. As just one of its many faults, the vertex contribution doesn't quite match the intuition: even in (2.3), based on the idea of which portion of the square lies in our polygon, it should be $1/16 = 0.625$ and not $\arctan(1/2)/2\pi = 0.073\ldots$.

From now on, we focus on the special case where P is an *integer polygon*, which means that all its vertices are integer points. This means that in (2.4) we are summing over all vertices. For any polygon, the sum of the interior angles at all the vertices is π(number of vertices -2). Therefore, we can rewrite the formula as follows:

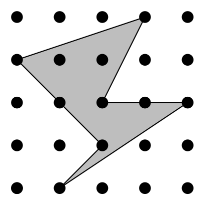
$$\begin{aligned} \text{area}(P) \approx\ & (\text{number of integer points in the interior of } P) \\ & + \tfrac{1}{2}(\text{number of integer points on the boundary of } P\text{, including vertices}) - 1. \end{aligned} \tag{2.5}$$

Now, a miracle happens:

Theorem 2.1. *(Pick's theorem) The formula* (2.5) *for integer polygons is an exact equality.*

Example 2.2. *The following polygon has* 2 *interior integer points, and* 8 *boundary integer points, hence area* 5*.*

(2.6)



(2b) Integer affine transformations. In classical geometry, congruences (Euclidean transformations) are key. When we talk about integer polygons, it's important to preserve integrality of coordinates. Among Euclidean transformations, this rules out almost all rotations and reflections, which is pretty poor. We propose a wider class of transformations:

Definition 2.3. *An* integer affine transformation *of the plane is a transformation of the form*

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto Av + w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} v + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ax+by+e \\ cx+dy+f \end{pmatrix} \tag{2.7}$$

where A is a 2×2 matrix with integer entries and $\det(A) = ad - bc = \pm 1$, and w is an integer vector.

Such transformations do not preserve lengths and angles, but they do preserve area, because the Jacobian has determinant 1. Most importantly for us, if v has integer coordinates, then so does $Av + w$, and vice versa. One can compose integer affine transformations:

$$v \longmapsto A_1 v + w_1 \longmapsto A_2(A_1 v + w_1) + w_2 = (A_2 A_1)v + (A_2 w_1 + w_2). \tag{2.8}$$

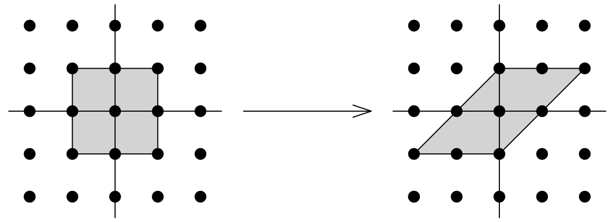
One can also reverse an integer affine transformation: the inverse of $v \mapsto Av + w$ is

$$v \longmapsto A^{-1}(v - w) = A^{-1}v + (-A^{-1}w), \tag{2.9}$$

and A^{-1} again has integer entries (by Cramer's rule).

Example 2.4. *The matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, with $w = 0$, gives the "shear" $(x, y) \mapsto (x + y, y)$:*

(2.10)



Definition 2.5. *Two integer polygons are called* integer affine equivalent *if there is an integer affine transformation carrying one to the other.*

Let's look at integer triangles T. After an integer translation, we can assume that the vertices are $(0,0)$, (a,c), and (b,d). The standard formula for area says that

$$\text{area}(T) = \tfrac{1}{2}\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|. \tag{2.11}$$

So the area is always a half-integer (this also follows from Pick's theorem, but I don't want to bother with that theorem here, since the triangle case is so explicit). Integer triangles with area $\frac{1}{2}$ are called minimal triangles.

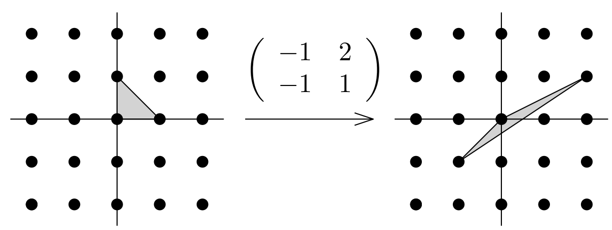
Fact 2.6. *Any two minimal integer triangles are integer affine equivalent.*

The reason is kind of staring us in the face. Take the "standard minimal triangle", which has vertices $(0,0)$, $(1,0)$, $(0,1)$. Then, any triangle with vertices $(0,0)$, (a,c) and (b,d) is the image of our "standard triangle" under the transformation

$$v \mapsto Av, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.12}$$

If the triangle is minimal, A has determinant ± 1 by the area formula, so this is an integer affine transformation. Here's an example:

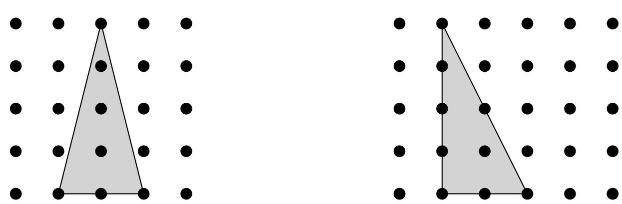
(2.13)



One can combine two such transformations to relate any two minimal triangles to each other. For integer triangles that are not minimal, there is no statement of that kind.

Example 2.7. *These two integer triangles*

(2.14)



both have area 4, but are not integer affine equivalent. This is easy to see: one has three points in its interior, the other only one; but integer affine transformations map integer points to integer points, and obviously interior points to interior points.

(2c) Pick's theorem by decomposition. How would one prove Pick's theorem? One way would be to gradually reduce it to simpler shapes.

FACT 2.8. *Take an integer polygon P and cut it into two integer polygonal pieces P_1, P_2 (for simplicity, let's say by a straight cut going from one integer boundary point of P to another). If Pick's theorem holds for P_1 and P_2, then it also holds for P.*

The cut creates a new edge. Any integer point on that edge, which is not an endpoint is counted as $1/2$ each in the count for P_1 and P_2, which matches the fact that it used to be an interior integer point for P, counted as 1. The two endpoints of the edge are boundary points of P, hence contribute $1/2 + 1/2 = 1$ to its count. After cutting, they contribute $1/2 + 1/2 = 1$ for each of P_1 and P_2, but that's compensated by subtracting 1 in (2.5).

FACT 2.9. *Pick's theorem holds for minimal triangles.*

Indeed, we know that any two minimal triangles are integer affine equivalent, so if it holds for any one of them (like our standard triangle), then it holds for all. This is not circular reasoning, we did not use Pick's theorem when talking about minimal triangles.

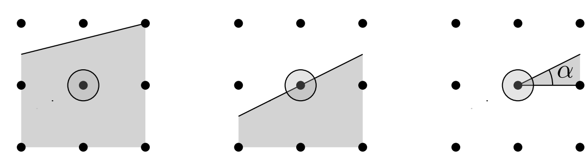
Then, to complete the proof of Pick's theorem, one would have to show that any integer polygon can, by repeated cutting, be divided into minimal triangles (which is true). Pick's theorem holds for each such triangle, and then by gradually putting the pieces back together, one gets it for the original polygon. But frankly, that method doesn't give a lot of intuition.

(2d) The oil spill argument. We want to outline a different proof of Pick's theorem, based on a thought experiment. At each integer point in the plane, we deposit a quantity (called 1) of oil. After that, the oil starts spreading out at speed 1, into a larger and larger circular drop of uniform density; and the drops merge, each continuing to spread as if the others didn't exist (you can think of each drop as lying on its own glass plate, and that we are looking at the whole parallel stack of plates from the top). Now introduce an integer polygon P, and ask: at a given time, how much oil is in P?

FACT 2.10. *A very small time after the oil is placed, the amount of oil in P is exactly the right hand side of Pick's formula* (2.5).

The idea is exactly as in (2.3), except that this time, we are exactly computing the amount of oil. Integer points make contributions like this:

(2.15)



1 unit of oil in P 1/2 oil in P $\alpha/2\pi$ oil in P

FACT 2.11. *As time passes, the amount of oil in P gets closer and closer to the area of P.*

Indeed, as the oil spreads, we get closer and closer to a uniform distribution of oil across the plane (with density 1, because we started with a unit of oil at each integer point).

FACT 2.12. *At any time, the net flow rate of oil across the boundary of P is zero.*

Let's forget about the whole of P, and just take one of its edges e (which is a line segment with integer endpoints). The claim is that the flow rate across e is zero. That is true for simple symmetry reasons. A 180 degree rotation around the midpoint of e preserves integer points, and therefore our oil picture at any time is unchanged under that rotation. But if the net flow across the edge was positive in one direction, after rotation, the rotated picture would show it to be positive in the other direction, which is a contradiction.

Because the net flow is zero, the quantity of oil contained in P doesn't change, which means that its value for small t (given by Fact 2.10) equals its limit as $t \to \infty$ (given by Fact 2.11). Bingo, Pick's theorem follows!