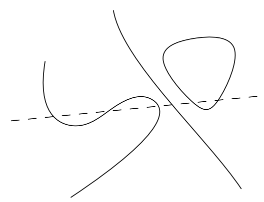
21. Intersections of algebraic curves

The structure of an algebraic curve is constrained by its degree. In this lecture we'll look at one aspect of this, namely how algebraic curves can intersect each other.

- We look at some simple (but still useful) cases, like the intersection of an algebraic curve and a line.
- Then we will state the general result, Bézout's theorem. Later, this will turn out to be a useful tool for studying the topology of algebraic curves.

(21a) Intersections with lines. The curve in the picture below could be algebraic, but if it is, its degree must be at least 6. To see that, one looks at the dashed line, and applies the following general observation:

(21.1)



Proposition 21.1. *Let C be a degree d curve, and L a line. Then C intersects L in at most d points, except in the case where L is actually a subset of C.*

Namely, take $C = \{f(x, y) = 0\}$, and parametrize the line L by $(x(t) = at + b, y(t) = ct + d)$. Points of C that lie on the line correspond to solutions of

$$f(x(t), y(t)) = f(at + b, ct + d) = 0, \tag{21.2}$$

which is a polynomial in one variable t of degree $\leq d$. That polynomial could be zero, in which case all t are solutions, and the line is contained in C. Otherwise, it is a basic fact that such a polynomial can only have at most d roots.

Proposition 21.2. *An algebraic curve of* odd *degree d can't be a bounded subset of the plane (it always goes out to infinity).*

We have $C = \{f(x, y) = 0\}$, where f has degree d. Suppose first that the x^d-coefficient of $f(x, y)$ is nonzero. Then, if we set y to be a constant, $f(x, y)$ is a polynomial in x of degree d. By another elementary fact, a polynomial of odd degree always has a root. This means that our curve intersects any horizontal line, and must therefore be unbounded.

Well then, what if the x^d-coefficient is zero? We can work around that by changing coordinates. Let's look at $f(x, cx + y)$, where c is some constant. If $f(x, y) = \sum_{i+j\leq d} a_{ij} x^i y^j$, then

$$\begin{aligned}(x^d\text{-coefficient of } f(x, cx + y)) &= a_{d,0} + a_{d-1,1}(x^d\text{-coefficient of } x^{d-1}(cx + y)) \\ &\quad + a_{d-2,2}(x^d\text{-coefficient of } x^{d-2}(cx + y)^2) + \cdots \\ &= a_{d,0} + a_{d-1,1}c + a_{d-2,2}c^2 + \cdots + c^d.\end{aligned} \tag{21.3}$$

Since f has degree d, one of the $a_{d-i,i}$ must be nonzero. Therefore, the expression (21.3) is a nonzero polynomial in c, and we can choose c so that the expression is not zero. Then, the previous argument applies after the coordinate change from (x, y) to $(x, cx + y)$.

(21b) Intersection with conics. Going back to the intersection problem, we look at the next case, that of conics.

PROPOSITION 21.3. *Let $C = \{f(x, y) = 0\}$ be a degree d curve, and D a conic. Then C intersects D in at most $2d$ points, with two exceptions. One exception is if D is contained in C. The second exception is if D is the union of two different lines, and one of those lines is a subset of C.*

One can prove this case-by-case by looking at the different kinds of conics. We'll do one example of each case:

- (Parabola) Suppose that $D = \{x^2 = y\}$, which we can parametrize by $(x(t), y(t)) = (t, t^2)$. Intersection points are solutions of $f(t, t^2) = 0$, which is a polynomial in t of degree $\leq 2d$, therefore has at most $2d$ roots.
- (Hyperbola) Suppose that $D = \{xy = 1\}$. If we set $(x(t), y(t)) = (t, t^{-1})$, then $f(t, t^{-1})$ is no longer a polynomial in t. Instead, it contains powers of t from t^{-d} to t^d. But if we multiply by t^d, we get a polynomial of degree $\leq 2d$, to which the previous argument applies.
- (Ellipse) As an example take the circle $D = \{x^2 + y^2 = 1\}$, for which we have the parametrization

$$x(t) = \frac{4t}{t^2 + 4}, \quad y(t) = \frac{t^2 - 4}{t^2 + 4}. \tag{21.4}$$

  If we insert that into the equation for $C = \{f(x, y) = 0\}$, we get a sum of terms

$$x(t)^i y(t)^j = \frac{(4t)^i (t^2 - 4)^j}{(t^2 + 4)^{i+j}} = \frac{(4t)^i (t^2 - 4)^j (t^2 + 4)^{d-i-j}}{(t^2 + 4)^d}. \tag{21.5}$$

  Therefore, $(t^2 + 4)^d f(x(t), y(t))$ is a polynomial of degree $\leq 2d$ in t. (This argument doesn't quite work if $(0, 1)$ lies on C, because our parametrization leaves out that point; but we can avoid that by rotating the coordinate plane before parametrizing.)
- (Other cases) D could consist of two lines, or one line, or a point, or is empty. All those are easy.

(21c) The general result. There is a theorem about intersections of algebraic curves of any degree, which includes all the cases discussed above. This is a much more difficult result, because curves of degree > 2 don't generally have rational parametrizations. As an introductory step, let's remind ourselves that if a polynomial can be written as a product of others,

$$f(x, y) = g(x, y)h(x, y), \tag{21.6}$$

then the curve $C = \{f(x, y) = 0\}$ is the union of $D = \{g(x, y) = 0\}$ and $E = \{h(x, y) = 0\}$:

$$C = D \cup E. \tag{21.7}$$

EXAMPLE 21.4. *Suppose that f has degree 3. Excluding the silly cases where g or h are constants, the way that (21.6) happens is that one of the factors (g, h) has degree 1 (and the other has degree 2). Then C contains a line. For all other degree 3 curves, f can't be factored into lower degree polynomials. For instance, $f(x,y) = x^3 + x - y^2$ can't be factored, since a quick look at the graph shows us that $\{f(x,y) = 0\}$ certainly doesn't contain a line.*

The general intersection problem is that we have two curves

$$C_1 = \{f_1(x,y) = 0\}, \quad C_2 = \{f_2(x,y) = 0\} \tag{21.8}$$

of degrees d_1 and d_2, respectively. The problem, as we already saw in the situations above, is that that there are exceptions: it is possible for $C_1 \cap C_2$ to be infinite, when the two curves have a part in common. Let's see how that could come about algebraically: suppose that f_1 and f_2 have a common factor g, which means they can be written as products of polynomials

$$\begin{aligned} f_1(x,y) &= g(x,y)h_1(x,y), \\ f_2(x,y) &= g(x,y)h_2(x,y), \end{aligned} \tag{21.9}$$

Then, every point where $g(x,y) = 0$ belongs to both C_1 and C_2. In terms of sets, take $D = \{g(x,y) = 0\}$, and $E_1 = \{h_1(x,y) = 0\}$, $E_2 = \{h_2(x,y) = 0\}$. We have

$$C_1 = D \cup E_1, \quad C_2 = D \cup E_2, \tag{21.10}$$

and therefore

$$C_1 \cap C_2 = D \cup (E_1 \cap E_2). \tag{21.11}$$

If D consists of infinitely many points, then the intersection $C_1 \cap C_2$ is clearly infinite. Bézout's theorem says that this is the only exception:

THEOREM 21.5. *(Bézout's theorem) Let $C_1 = \{f_1(x,y) = 0\}$ and $C_2 = \{f_2(x,y) = 0\}$ be algebraic curves of degrees d_1 and d_2. Then, C_1 intersects C_2 in at most $d_1 d_2$ points, except in the following situation: (21.9) holds, where $g(x,y)$ is such that $D = \{g(x,y) = 0\}$ has infinitely many points, and h_1, h_2 have no common factor (the last one is clear because we can move any common factor to g).*

In the "except" situation, one can apply Bézout's theorem another time, to (h_1, h_2) (which have no common factor) to fully describe $C_1 \cap C_2$.

EXAMPLE 21.6. *Any curve of degree d intersects the curve $\{x^3 + x - y^2 = 0\}$ in at most $3d$ points. In that case, the exceptional situation is impossible, because the degree 3 polynomial can't be factored.*

EXAMPLE 21.7. *Suppose that C_1 and C_2 both have degree 3. Then, the cases break down as follows.*

- (21.9) *applies with g of degree 3. This means that h_1 and h_2 are constants, so f_1 and f_2 are multiples of each other: $C_1 = C_2$.*

- (21.9) *applies with* g *of degree* 2. *Then* h_1, h_2 *are of degree* 1, *and have no common factor, so they are different lines. This means that* $E_1 = \{h_1 = 0\}$ *and* $E_2 = \{h_2 = 0\}$*: the intersection* $E_1 \cap E_2$ *is empty or a single point. In the end,* $C_1 \cap C_2 = D \cup (E_1 \cap E_2)$ *consists of a conic* D *(which has infinitely many points) and at most one additional point.*
- (21.9) *applies with* g *of degree* 1, *so* $D = \{g(x, y) = 0\}$ *is a line. Then* h_1, h_2 *are of degree* 2, *and have no common factor. We can apply Bézout to* (h_1, h_2), *and find that* $E_1 \cap E_2$ *is at most four points. So,* $C_1 \cap C_2$ *consists of a line* D *and at most four additional points.*
- *Finally, there's the case where the "main branch" of Bézout applies, meaning* $C_1 \cap C_2$ *consists of* ≤ 9 *points (this is what happens most of the time).*