22. Nonsingular curves

Many, but not all, algebraic curves look like they are smoothly embedded (no kinks or selfintersections). Formally, these are the *nonsingular curves.*

- Topologically, a nonsingular curve consists of bounded components (which look like loops, called ovals) and unbounded ones (curves going to infinity at both ends).
- We discuss a number of basic results concerning the number and position of ovals, depending of course on the degree of the curve.

(22a) Nonsingular points. The circle $x^2 + y^2 = 1$ is smooth, in a way in which the curves $xy = 0$ or $x^3 + y^2 = 0$ are not. The mathematics behind that is:

Definition 22.1. *A solution (x, y) of $f(x, y) = 0$ is called a singular point if the partial derivatives $\partial_x f = \frac{\partial f}{\partial x}$ and $\partial_y f = \frac{\partial f}{\partial y}$ are both zero at that point. Otherwise, it is called nonsingular. We say that the equation $f(x, y) = 0$ is nonsingular if all of its solutions are nonsingular.*

Often, we will say "C is a nonsingular curve of degree d". By this we mean that there is a degree d polynomial $f(x, y)$ such that the equation $f(x, y) = 0$ is nonsingular and defines the curve C.

Example 22.2. *Take $f(x, y) = p(x) - y^2$. Then*

$$\begin{aligned} \partial_x f &= p'(x), \\ \partial_y f &= -2y. \end{aligned} \tag{22.1}$$

Therefore, a singular point is of the form $(x, 0)$, where $p(x) = 0$ and $p'(x) = 0$. The equation $f(x, y) = 0$ is nonsingular if and only if there are no such points.

Example 22.3. *Take $f(x, y) = f_1(x, y)f_2(x, y)$. Then: any singular point of $f_1(x, y) = 0$ is a singular point of $f(x, y) = 0$, and the same is true for $f_2(x, y) = 0$. Moreover: any point (x, y) where both $f_1(x, y) = 0$ and $f_2(x, y) = 0$ becomes a singular point of $f(x, y) = 0$. This is easy to see: the product rule says that*

$$\begin{aligned} \partial_x(f_1 f_2) &= (\partial_x f_1)f_2 + f_1(\partial_x f_2), \\ \partial_y(f_1 f_2) &= (\partial_y f_1)f_2 + f_1(\partial_y f_2). \end{aligned} \tag{22.2}$$

If (x, y) is a singular point of $f_1(x, y) = 0$, then f inherits the vanishing of derivatives, so it becomes a singular point of $f(x, y) = 0$. The same holds for $f_2(x, y) = 0$. Moreover, if both f_1 and f_2 are zero at (x, y), then that also causes the partial derivatives (22.2) *to vanish.*

Theorem 22.4. *If $f(x, y) = 0$ is nonsingular, the curve $C = \{f(x, y) = 0\}$ is a disjoint union of components of two kinds: bounded components, also called* ovals*, each of which can be traced out by an embedded loop; and unbounded components, which are embedded curves going off to infinity at both ends. On the two sides of any component, $f(x, y)$ has opposite signs.*

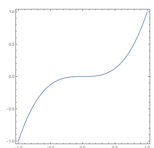
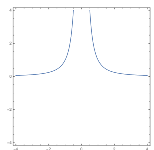
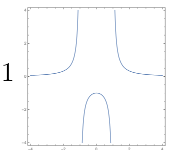
The main topological question is, how many components can there be, and arranged how?

Example 22.5. *An equation $f(x, y) = 0$ of degree 2 is nonsingular exactly if the resulting curve is one of the following:*

- *an ellipse (one oval);*
- *a parabola (one unbounded component);*
- *a hyperbola (two unbounded components);*
- *two parallel lines (two unbounded components);*
- *the empty set (which satisfies "all points are nonsingular points of $f(x, y) = 0$" in the same sense as "all my Ferraris are green").*

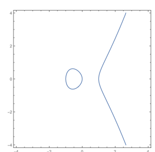
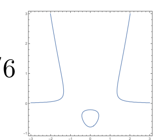
(22b) Degree 3 curves. A nonsingular curve of degree 3 must have at least one unbounded component; otherwise, it would be contained in a bounded subset of the plane, and we saw in the previous lecture that this is impossible.

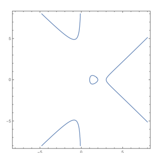
EXAMPLE 22.6. *The curves below have one, two, or three unbounded components, and zero ovals:*

(22.3) $y = x^3$  $x^2 y = 1$  $y(x^2 - 1) = 1$ 

These have one oval, and one, two or three unbounded components:

(22.4)

$y^2 = x^3 - x$  $y(y - x^2 + 1) = -1/6$ 

$y^2 x = (x - 1)(x - 2)(x - 3)$ 

It will turn out that these are the only possibilities!

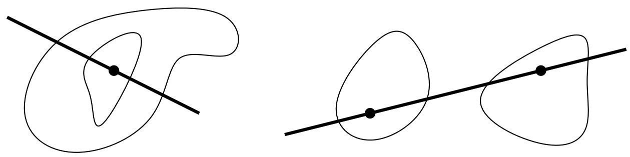
Like any embedded loop, an oval divides the plane into a bounded (inside) and unbounded (outside) region. Therefore, any line through a point inside the oval must intersect that oval at least twice (since it goes towards infinity at both ends). One can use that and Bézout's theorem to show the following:

PROPOSITION 22.7. *A nonsingular algebraic curve of degree* 3 *has at most one oval.*

Let's separate out one case, which is where the defining equation of our curve is a product of two polynomials, of degrees 1 and 2. Because of nonsingularity, the zero-sets of those two polynomials can't intersect (see Example 22.3), and the desired property is easy to see from the structure of lines and conics.

The main work goes into the other case, where the defining equation does not factor. We argue by contradiction: suppose that there are two ovals. They could be nested one inside the other, or not. In the nested case, we take a point lying inside the innermost oval. A line through that point must intersect each oval twice, yielding a total of four intersection points with the algebraic curve. But that's impossible, by Bézout's theorem. Similarly, if the two ovals are not nested, we take one point inside each, and connect those two points by a line, with the same outcome.

(22.5)



Proposition 22.8. *A nonsingular algebraic curve of degree* 3 *has at most three unbounded components.*

Let's just consider the case where the defining equation is not a product. Each unbounded component goes out to infinity at both its ends. So, if we take a sufficiently large circle, the unbounded component will intersect that circle at least twice. If there are four unbounded components, this would give at least eight intersection points with a large circle, again contradicting Bézout's theorem.

(22c) Higher degrees. For the general discussion, we'll focus on the ovals. We'll need to know this:

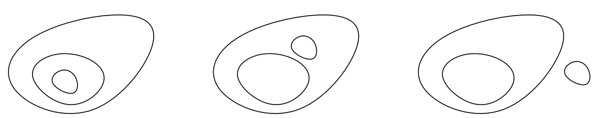
Lemma 22.9. *Suppose that C is a nonsingular algebraic curve. Take an oval $O \subset C$, a point p inside that oval, and another point q outside. If a conic goes through both p and q, it must intersect O at least twice.*

If our conic is unbounded (a line, a union of two lines, a parabola, or a hyperbola), it's enough to have one point inside the oval; the previously used argument applies. This leaves the case of ellipses, where we use the fact that we have to travel from inside the oval to outside and then back.

Proposition 22.10. *A nonsingular algebraic curve of degree* 4 *can have at most* 4 *ovals. Moreover, if it has* 2 *ovals nested inside each other, then it can't have any other ovals.*

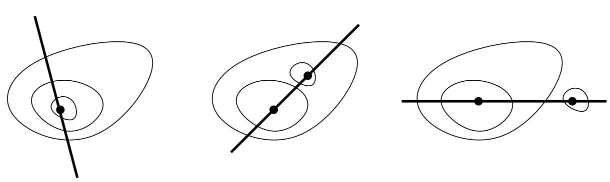
Let's just consider the case where the defining polynomial is not a product. Suppose that we have two ovals nested inside each other, plus another oval. There are three possibilities for how the three ovals could be arranged:

(22.6)



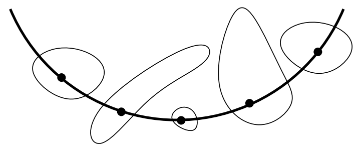
All of those possibilities can be ruled out by a judicious choice of line, which produces more than 4 intersection points, hence violates the Bézout bound:

(22.7)



Similarly, suppose that we have at least 5 ovals, none of them nested inside each other. Pick a point inside each oval. We now appeal to interpolation (Lemma 19.5): there is a conic which goes through all 5 points. This conic will intersect each oval at least twice, by Lemma 22.9, so one gets at least 10 intersection points of the conic and the curve, which contradicts Bézout's theorem.

(22.8)



THEOREM 22.11. *(Harnack's theorem for the Euclidean plane) A nonsingular algebraic curve of degree d can have at most M ovals, where*

$$M = \begin{cases} \frac{d(d-3)}{2} + 2 & \text{if } d \text{ is even,} \\ \frac{d(d-3)}{2} + 1 & \text{if } d \text{ is odd.} \end{cases} \tag{22.9}$$

The proof follows the same idea as the $d = 3, 4$ cases, of constructing an auxiliary curve and applying Bézout's theorem.

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