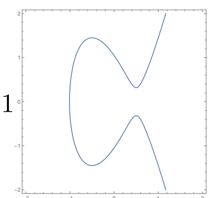
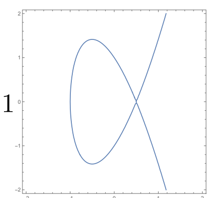
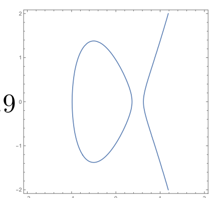
23. Singular points

Even though a randomly chosen algebraic curve will be nonsingular, the singular curves play a particular role in the theory.

- We will introduce the simplest class of singular points, called nodes (they are essentially the local minima, local maxima, and saddle points for functions of two variables, which are familiar from multivariable calculus).
- By slightly tweaking the defining polynomial, one can remove a node. This turns out to be a good way of constructing algebraic curves with interesting topological structure.

(23a) Nodes. Take $4x^3 - 3x - y^2 = b$, where $b > 0$ is a parameter:

(23.1) $b = -1.1$  $b = -1$  $b = -0.9$ 

The $b = -1$ case has a singular point. At that value, the structure of our curves changes: the $b > -1$ curves have an oval, but the $b < -1$ curves don't. This shows the importance of singular points for understanding the topology even of nonsingular curves.

The general setup is this. Given $C = \{f(x,y) = 0\}$, recall that a point (x_0, y_0) of C is a singular point if both f and its derivatives are zero at that point:

$$f(x_0, y_0) = 0, \quad (\nabla f)_{(x_0,y_0)} = 0. \tag{23.2}$$

The next step is to look at the symmetric matrix of second derivatives, the Hessian

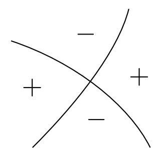
$$\mathrm{Hess}(f) = \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_y \partial_x f & \partial_y^2 f \end{pmatrix}. \tag{23.3}$$

Definition 23.1. *A singular point is called a* node *if the Hessian $H = \mathrm{Hess}(f)(x_0, y_0)$ at that point has nonzero determinant. As you may remember from multivariable calculus, there are three sub-cases:*

- *If $\det(H) > 0$ and the trace is $\mathrm{tr}(H) > 0$, the point (x_0, y_0) is a local minimum of f. In that case, (x_0, y_0) sits by itself on C (there are no other points nearby, since f becomes positive).*
- *If $\det(H) > 0$ and $\mathrm{tr}(H) < 0$, the point (x_0, y_0) is a local maximum of $f(x,y)$. Again, (x_0, y_0) sits by itself on C (the only difference being that f becomes negative nearby).*
- *If $\det(H) < 0$, (x_0, y_0) is a saddle point. In this case, the local picture of C is that two branches cross transversally. The four nearby regions of the plane have different signs*

of f, like this:

(23.4)



Example 23.2. *The $b = -1$ case of (23.1) has a node, which is a saddle point, at $(x_0, y_0) = (1/2, 0)$. To check that this is the case, we take $f(x,y) = 4x^3 - 3x - y^2 + 1$, and compute*

$$\nabla f = \begin{pmatrix} 12x^2 - 3 \\ -2y \end{pmatrix}, \quad \det(\mathrm{Hess}(f)) = -48x. \tag{23.5}$$

We have $f(x_0, y_0) = 0$ (the point lies on the curve), $(\nabla f)_{(x_0,y_0)} = 0$ (it's a singular point), and $\det(\mathrm{Hess}(f)_{(x_0,y_0)}) = -24 < 0$ (it's a saddle point).

Lemma 23.3. *Take polynomials $f_1(x,y)$ and $f_2(x,y)$. Let (x_0, y_0) be a solution both of $f_1(x_0, y_0) = 0$ and $f_2(x_0, y_0) = 0$ (geometrically, it's an intersection point of the resulting algebraic curves). Suppose that the gradient vectors ∇f_1, ∇f_2 at (x_0, y_0) are linearly independent. Then:*

- *for $f(x,y) = f_1(x,y) f_2(x,y)$, the equation $f(x,y) = 0$ has a node at (x_0, y_0), which is a saddle point.*
- *for $g(x,y) = f_1(x,y)^2 + f_2(x,y)^2$, the equation $g(x,y) = 0$ has a node at (x_0, y_0), which is a local minimum.*

That's geometrically intuitive, but one can also check it by explicit computation: the Hessian of f at (x_0, y_0) satisfies $\det(\mathrm{Hess}(f)) = -(\nabla f_1 \times \nabla f_2)^2 < 0$. For the Hessian of g, one gets $\det(\mathrm{Hess}(g)) = 2(\nabla f_1 \times \nabla f_2)^2 > 0$.

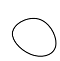
(23b) Perturbing nodes. What happens if, starting with an algebraic curve with a node, one slightly modifies the defining polynomial? Suppose that $C = \{f(x,y) = 0\}$ has a node at (x_0, y_0). Let $g(x,y)$ be another polynomial such that $g(x_0, y_0) \neq 0$, and look at

$$\tilde{C} = \{f(x,y) = \epsilon g(x,y)\}, \tag{23.6}$$

where ϵ is a (sufficiently) small nonzero parameter.

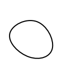
- If f has a local minimum at (x_0, y_0), depending on the signs, the singular point will either disappear or be replaced by a small oval:

(23.7)

nothing	$\bullet$	
$f(x,y) = \epsilon\, g(x,y)$ $\epsilon\, g(x_0,y_0) < 0$	$f(x,y) = 0$	$f(x,y) = \epsilon\, g(x,y)$ $\epsilon\, g(x_0,y_0) > 0$

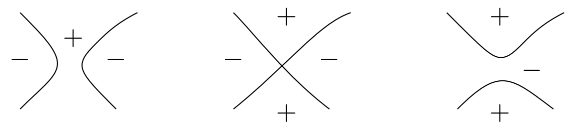
- If f has a local maximum at (x_0, y_0), the same happens, with switched signs:

(23.8)

	$\bullet$	nothing
$f(x,y) = \epsilon\, g(x,y)$ $\epsilon\, g(x_0,y_0) < 0$	$f(x,y) = 0$	$f(x,y) = \epsilon\, g(x,y)$ $\epsilon\, g(x_0,y_0) > 0$

- If f has a saddle point at (x_0, y_0), two of the neighbouring regions will merge:

(23.9)



$f(x,y) = \epsilon\, g(x,y)$, $\epsilon\, g(x_0,y_0) < 0$ $\qquad$ $f(x,y) = 0$ $\qquad$ $f(x,y) = \epsilon\, g(x,y)$, $\epsilon\, g(x_0,y_0) > 0$

To understand this, one studies quadratic models, say $f(x, y) = xy$ or $x^2 + y^2$ or $-x^2 - y^2$, and takes $g(x, y)$ to be a nonzero constant. This means that all we are doing is looking at the level sets of these quadratic functions. It's a good model for our problem because the Hessian gives the quadratic approximation near the singular point.

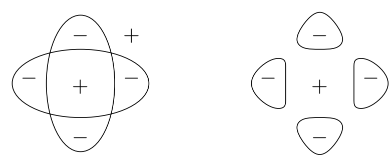
(23c) Applications. We can use this idea to construct algebraic curves with different shapes.

Example 23.4. *Start with the union of two intersecting ellipses,*

$$\begin{aligned} &f(x,y) = f_1(x,y) f_2(x,y), \\ &f_1(x,y) = x^2 + 4y^2 - 1,\ f_2(x,y) = 4x^2 + y^2 - 1. \end{aligned} \tag{23.10}$$

By perturbing this, we get a nonsingular degree 4 *curve with* 4 *ovals (which is the maximal number allowed by Harnack's theorem):*

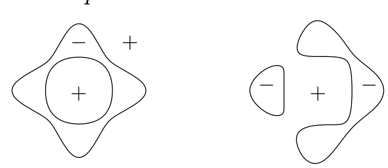
(23.11)



$C = \{f(x,y) = 0\}$ $\qquad$ $\tilde{C} = \{f(x,y) = \epsilon\}$, $\epsilon < 0$

There are a number of other possible perturbations:

(23.12)



$\tilde{C} = \{f(x,y) = \epsilon\}$, $\epsilon > 0$ $\qquad$ $\tilde{C} = \{f(x,y) = \epsilon x\}$, $\epsilon > 0$

Example 23.5. *Take f_1, f_2 as in the previous example, but now*

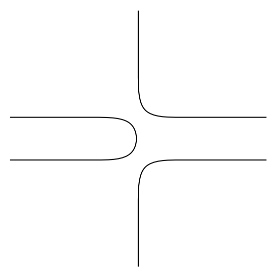
$$f(x,y) = f_1(x,y)^2 + f_2(x,y)^2. \tag{23.13}$$

This consists of just 4 *points, but if we look at $f(x, y) = \epsilon$ for some small $\epsilon > 0$, each point expands into a small oval. This gives another construction of a degree* 4 *curve with* 4 *ovals.*

Finally, a warning is appropriate, which we have glossed over in the examples above. When perturbing to $f(x, y) = \epsilon g(x, y)$, we have explained what happens to nodes. Around a nonsingular point, the curve will move only a little (for sufficiently small ϵ), without changing its shape. However, but close to infinity (far out in the plane), it is possible that more drastic changes might happen. In practice this means that you can control what happens in any given bounded subset of the plane (for instance, to produce a certain desired arrangement of ovals). However, if you want full information about the curve you've produced, you should at least look at a computer plot, to check that no undesirable effects have occurred near infinity (there are more rigorous arguments, but they're beyond our scope here). This is not a dramatic failure, since I think of this more as a discovery method; after all, we haven't specified what "small ϵ" means quantitatively either.

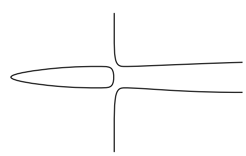
EXAMPLE 23.6. *Take $f(x,y) = y(y-1)x$, which consists of two vertical lines and a horizontal line. This has two saddle point singularities. Take $g(x,y) = x^2+1$, and look at $C = \{f(x,y) = \epsilon g(x,y)\}$ for small $\epsilon > 0$. From just thinking of the nodes, one might expect C to look like this:*

(23.14)



and that's what happens near the nodes. However, if you zoom out far enough, you'll see that the overall shape of the curve looks like this instead:

(23.15)



By accident, while removing the nodes, we have also perturbed our two parallel lines to a (very thin) parabola!

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Spring 2023