24. Patchworking

Patchworking (invented by Viro) is another way of constructing nonsingular algebraic curves with prescribed topology. It involves polynomials whose coefficients are of vastly different sizes.

- The process is combinatorial and, in its basic version, very easy to carry out.
- It is another question altogether why it works; we'll only be able to give you some hints.

(24a) Patchworking. We look at polynomials in a very specific form, which depend on a parameter $t > 0$, thought of as small:

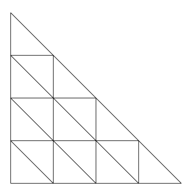
$$f_t(x,y) = \sum_{i+j \leq d} \sigma_{ij} t^{w_{ij}} x^i y^j. \tag{24.1}$$

Here, $\sigma_{ij} \in \{\pm 1\}$ are signs that we can choose freely, and the powers of t are prescribed:

$$w_{ij} = \frac{i(i-1)}{2} + \frac{j(j-1)}{2} + \frac{(i+j)(i+j-1)}{2}. \tag{24.2}$$

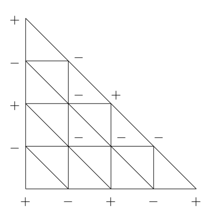
Take the triangle T_d with vertices $(0,0)$, $(d,0)$, $(0,d)$; and decompose it into d^2 smaller triangles, in the following specific way:

(24.3)



Each integer point (i,j) in T_d represents a monomial in f_t, and we mark it with the corresponding sign σ_{ij}. The markings below

(24.4)

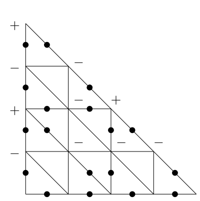


correspond to

$$\begin{aligned} f_t(x,y) = 1 - x - y + t^2x^2 - txy + t^2y^2 - t^6x^3 - t^4x^2y - t^4xy^2 - t^6y^3 \\ + t^{12}x^4 - t^9x^3y + t^8x^2y^2 - t^9xy^3 + t^{12}y^4. \end{aligned} \tag{24.5}$$

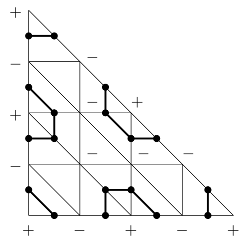
In the next step, if an edge of one of the small triangles connects two vertices with opposite signs, we mark a point on that edge:

(24.6)



Each of the small triangles has an even number (either 0 or 2) of edges with marked points. If there are 2 such marked points, we connect them by a line inside the small triangle:

(24.7)



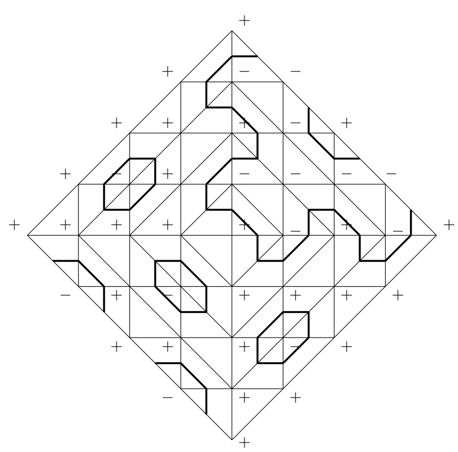
The outcome is a “topologically correct” picture of $\{f_t(x,y) = 0,\ x, y > 0\}$, assuming $t > 0$ is chosen sufficiently small (I can't tell you how small, but as degrees get higher, this will need to be really tiny).

To capture the entire zero-set of $f_t(x,y)$, one applies the previous process to $f_t(\pm x, \pm y)$. Pictorially, it is convenient to reflect the original triangle and its decomposition along the coordinate axes, forming a diamond shape. To each integer point in that shape, one associates a sign, by starting with the original signs in the triangle, and applying the following rules.

- when reflecting along the vertical axis, reverse the signs of points (i,j) with odd i; and
- when reflecting along the horizontal axis, reverse the signs of points (i,j) with odd j.

This just expresses which monomials $x^i y^j$ change signs under $(x,y) \mapsto (-x,y)$ or $(x,y) \mapsto (x,-y)$. Finally, one draws lines as before:

(24.8)



We can now be a bit more explicit about what “topologically correct” means: the algebraic curve we are looking at is nonsingular, and the picture correctly describes the topology of each of its components (oval or unbounded), as well as how they are arranged with respect to each other. Of course, it represents those components as stick figure caricatures, but that's irrelevant. In our running example, the outcome is that $f_t(x,y) = 0$, for small $t > 0$, is a (nonsingular degree 4) curve with three ovals (not nested inside each other) and 4 unbounded components.

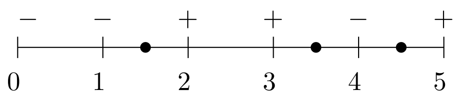
(24b) A one-variable analogue. Let's look at polynomials in one variable, again with a parameter $t > 0$, of a special form. Namely, we take $\sigma_i \in \{\pm 1\}$, $w_i = i(i-1)/2$, and consider

$$p_t(x) = \sigma_0 t^{w_0} + \sigma_1 t^{w_1} + \cdots + \sigma_d t^{w_d} = \sigma_0 + \sigma_1 x + \sigma_2 t x + \sigma_3 t^3 x^3 + \cdots \tag{24.9}$$

Proposition 24.1. *For small t, p_t has as many positive roots (solutions of $p_t(x) = 0$ with $x > 0$) as there are sign changes in the sequence $(\sigma_0, \dots, \sigma_d)$. More precisely, to each sign change $\sigma_i \neq \sigma_{i+1}$ corresponds a root $x \approx t^{-i}$.*

The similarity with patchworking becomes evident when we represent this graphically. Draw $[0, d]$ subdivided into unit intervals, with the sign σ_i associated to the point i, and then insert a dot between σ_i and σ_{i+1} whenever their signs are opposite. Those dots represent the positive zeros of our polynomial. For instance, take

(24.10)



$$p_t(x) = -1 - x + t x^2 + t^3 x^3 - t^6 x^4 + t^{10} x^5$$

In this example, Proposition 24.1 says that there should be three positive roots, at $x_1 \approx t^{-1}$, $x_2 \approx t^{-3}$, $x_3 \approx t^{-4}$. Let's check this against the actual location of the positive roots (determined numerically, hence approximate):

t	x_1	x_2	x_3
10^{-1}	$1.001 \cdot 10^1$	$1.232 \cdot 10^3$	$0.886 \cdot 10^4$
10^{-2}	$1.000 \cdot 10^2$	$1.020 \cdot 10^6$	$0.990 \cdot 10^8$
10^{-3}	$1.000 \cdot 10^3$	$1.002 \cdot 10^9$	$0.999 \cdot 10^{12}$

The advantage of the one-variable situation is that we can explain this phenomenon with a minimum of fuss. Let's look at our polynomial with the x-variable rescaled in a t-dependent way:

$$\begin{aligned} & p_t(x) \approx \sigma_0 + \sigma_1 x, \\ & (x^{-1} t)\, p_t(t^{-1} x) \approx \sigma_1 + \sigma_2 x, \\ & \dots \\ & (x^{-i} t^{i(i+1)/2})\, p_t(t^{-i} x) \approx \sigma_i + \sigma_{i+1} x. \end{aligned} \tag{24.11}$$

Here, $\approx$ means that we neglect terms with positive powers of t (thinking about the limit $t \to 0$, in which those become zero). If $\sigma_0 \neq \sigma_1$, then $\sigma_0 + \sigma_1 x = 0$ has the obvious solution $x = 1$. In view of (24.11), one can then find a root of $p_t(x)$, for small t, with $x \approx 1$. Similarly, if $\sigma_i \neq \sigma_{i+1}$, then $\sigma_i + \sigma_{i+1} x = 0$ has the solution $x = 1$, and therefore $p_t(t^{-i} x) = 0$ has a solution $x \approx 1$, which means that $p_t(x) = 0$ has a solution $x \approx t^{-i}$. This explains most of Proposition 24.1: that there are at least as many positive roots as there are sign-changes, and the approximate position of those roots. The rest, that there are no other positive roots, can be derived from a classical theorem (Descartes' rule of signs); we won't discuss that here.

The idea behind patchworking is the same: after a suitable t-dependent rescaling of (x, y)-coordinates, one can approximate the polynomial by one with only three terms, whose zero-set is a straight line. The union of those lines then gives a picture of $\{f_t(x, y) = 0\}$. While the details are not as easy as in the one-variable case, what you should take away from this is that the pieces of the patchworked curve actually live at quite different scales in the (x, y)-coordinates (making ordinary graphing software totally ineffective).

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