25. Tropical geometry

On the face of it, tropical geometry is a new kind of algebraic geometry, where we replace the arithmetic operations by different and simpler ones. It is closely related to patchworking, for which it provides a more quantitatively accurate viewpoint.

- We introduce tropical arithmetic, tropical polynomials, and tropical algebraic curves.
- We relate them to ordinary arithmetic and ordinary algebraic curves, by taking logarithms with respect to large bases.

(25a) The tropical numbers. The tropical numbers are

$$\mathbb{R}_{\mathrm{trop}} = \{-\infty\} \cup \mathbb{R}, \tag{25.1}$$

but with different addition and multiplication operations:

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \odot b &= a + b. \end{aligned} \tag{25.2}$$

Since tropical multiplication is ordinary addition, the tropical multiplicative neutral element is the ordinary additive neutral element, meaning 0. The tropical additive neutral element is $-\infty$. Not all of the usual rules hold: there can't be tropical subtraction, since it's impossible to recover a from knowing b and $\max(a, b)$.

One can relate tropical numbers to ordinary nonnegative real numbers as follows. Fix some $s > 0$, thought of as being large. The correspondence is

$$\begin{aligned} a \in \mathbb{R}_{\mathrm{trop}} &\longrightarrow x = s^a \in \mathbb{R}^{\geq 0}, \\ a = \log_s(x) \in \mathbb{R}_{\mathrm{trop}} &\longleftarrow x \in \mathbb{R}^{\geq 0}, \end{aligned} \tag{25.3}$$

with the convention that $s^{-\infty} = 0$ and $\log_s(0) = -\infty$. This is compatible with multiplications,

$$\log_s(s^a \cdot s^b) = a + b = a \odot b. \tag{25.4}$$

It is also approximately compatible with additions, with an error that goes to 0 as $s \to \infty$:

Lemma 25.1. *For $a, b \in \mathbb{R}^{\mathrm{trop}}$,*

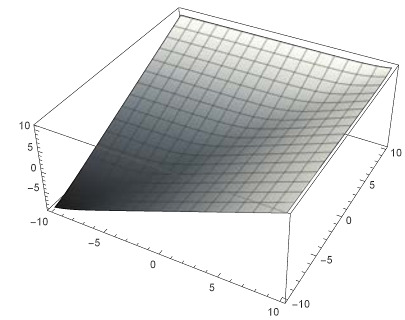
$$a \oplus b \leq \log_s(s^a + s^b) \leq a \oplus b + \frac{1}{s^{|a-b|} \ln(s)}. \tag{25.5}$$

The first inequality in (25.5) follows from the fact that $\log_s$ is an increasing function. For the other one, it's enough to look at the case $a \geq b$. Using the fact that $\ln(1 + c) \leq c$, we then write

$$\begin{aligned} \log_s(s^a + s^b) &= \log_s(s^a(1 + s^{b-a})) = \log_s(s^a) + \log_s(1 + s^{b-a}) \\ &= a + \frac{\ln(1 + s^{b-a})}{\ln(s)} \leq a + \frac{1}{s^{a-b} \ln(s)}. \end{aligned} \tag{25.6}$$

Visually, it's eyecatching how the graph of $\ln(e^a + e^b)$ bends to approximate $\max(a, b)$:

(25.7)



(25b) Tropicalization of polynomials. Take a polynomial in two variables, depending on an additional parameter s, of the form

$$f_s(x, y) = \sum_{i+j \leq d} s^{-w_{ij}} x^i y^j, \quad w_{ij} \in \mathbb{Z}. \tag{25.8}$$

To find the tropical analogue of f_s, we replace: x with $a = \log_s(x)$; y with $b = \log_s(y)$; the powers $s^{-w_{ij}}$ with the constants $\log_s(s^{-w_{ij}}) = -w_{ij}$; and all arithmetic operations with their tropical counterparts:

$$f_{\mathrm{trop}}(a, b) = \bigoplus_{i+j \leq d} \big((-w_{ij}) \odot \overbrace{a \odot \cdots \odot a}^{i \text{ terms}} \odot \overbrace{b \odot \cdots \odot b}^{j \text{ terms}}\big). \tag{25.9}$$

In more concrete terms, this is a piecewise linear function:

$$f_{\mathrm{trop}}(a, b) = \max_{i+j \leq d}\{ia + jb - w_{ij}\}. \tag{25.10}$$

Example 25.2. *If $f_s(x, y) = 1 + x + s^{-3}x^2y^2$, then $f_{\mathrm{trop}}(a, b) = \max\{0, a, 2a + 2b - 3\}$.*

The tropical version of $f_s(x, y) = 0$ is $f_{\mathrm{trop}}(a, b) = -\infty$ (since $-\infty$ is the additive unit in the tropical numbers). However, that's not particularly interesting in either context: $f_s(x, y) = 0$ has no solutions with $x, y > 0$, and correspondingly $f_{\mathrm{trop}}(a, b) = -\infty$ has no solutions with $a, b > -\infty$. We therefore look at polynomials which have terms of either sign:

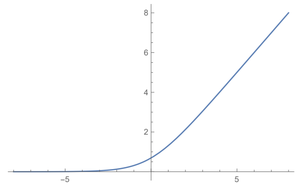
$$f_s(x, y) = \sum_{i+j \leq d} \sigma_{ij} s^{-w_{ij}} x^i y^j, \quad w_{ij} \in \mathbb{Z}, \ \sigma_{ij} \in \{\pm 1\}. \tag{25.11}$$

To tropicalize the equation $f_s(x, y) = 0$, we separate the polynomial into positive and negative terms, $f_s(x, y) = f_s^+(x, y) - f_s^-(x, y)$. Then, the algebraic curve associated to f_s can be written without subtraction as $C_s = \{f_s^+(x, y) = f_s^-(x, y)\}$. Its tropicalization is accordingly

$$C_{\mathrm{trop}} = \{(a, b) \ : \ f^+_{\mathrm{trop}}(a, b) = f^-_{\mathrm{trop}}(a, b)\}. \tag{25.12}$$

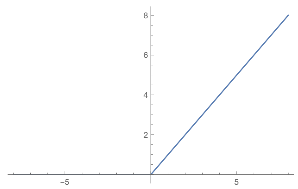
Example 25.3. *Take $C_s = \{1 + x - y = 0\} = \{y = x + 1\}$, which in this particular case is independent of s. In s-dependent coordinates $x = s^a$, $y = s^b$, we get $C_s = \{b = \log_s(s^a + 1)\}$,* *which looks like this:*

(25.13)



Compare this with $C_{\text{trop}} = \{b = \max(a, 0)\}$*:*

(25.14)



(25c) Tropical patchworking. Generally speaking, drawing C_{trop} can be quite complicated, as one has to figure out which of the many terms is the maximal one in $f^{\pm}_{\text{trop}}(a, b)$ for any point (a, b). The situation becomes simpler if we take the exponents of s to be those from our previous discussion of patchworking,

$$w_{ij} = \frac{i(i-1)}{2} + \frac{j(j-1)}{2} + \frac{(i+j)(i+j-1)}{2}. \tag{25.15}$$

In this case, the computation of maxima simplifies, leaving the following contributions to C_{trop}:

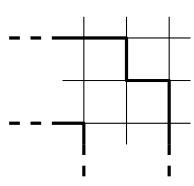
- for each (i, j) such that $\sigma_{i+1,j} \neq \sigma_{i,j}$, we get a piece of the vertical line $a = 2i + j$, which is the solution set of $ai + bj + w_{ij} = a(i+1) + bj + w_{i+1,j}$;
- for each (i, j) such that $\sigma_{i,j+1} \neq \sigma_{i,j}$, we get a piece of the horizontal line $b = i + 2j$, which is the solution set of $ai + bj + w_{ij} = ai + b(j+1) + w_{i,j+1}$;
- for each (i, j) such that $\sigma_{i+1,j-1} \neq \sigma_{i,j}$, we get a piece of the diagonal line $b - a = j - i - 1$, which is the solution set of $ai + bj + w_{ij} = (a+1)i + (b-1)j + w_{i+1,j-1}$.

It's useful to look at an example. Let's take a fairly simple one,

$$f_s(x, y) = 1 - x - y + s^{-2}x^2 + s^{-1}xy + s^{-2}y^2. \tag{25.16}$$

Here is the picture of all the lines listed above, with the actual C_{trop} marked in bold:

(25.17)

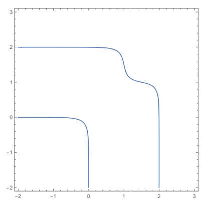


We can compare it with the actual algebraic curve in log coordinates. Let's introduce the notation

$$\begin{aligned} &\mathrm{Log}_s : (\mathbb{R}^{\geq 0})^2 \longrightarrow (\{-\infty\} \cup \mathbb{R})^2, \\ &\mathrm{Log}_s(x, y) = (\log_s(x), \log_s(y)). \end{aligned} \tag{25.18}$$

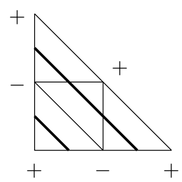
Then, $\mathrm{Log}_s(C_s \cap (\mathbb{R}^{\geq 0})^2)$ looks like this (for s large, in this case $s = 100$):

(25.19)



Moreover, if we set $s = t^{-1}$, then C_t is an example of patchworking according to this diagram:

(25.20)



What this example reveals is actually part of a general pattern. C_{trop} is a modified version of the patchworking diagram; both are essentially combinatorial (stick-figure) objects, and one can go back and forth between them, without affecting the qualitative (topological) structure. Secondly, as one would guess from our discussion of the relation between ordinary and tropical numbers:

Theorem 25.4. *In the situation of* (25.15)*, we have that as* $s \to \infty$*,*

$$\mathrm{Log}_s(C_s \cap (\mathbb{R}^{\geq 0})^2) \longrightarrow C_{\mathrm{trop}}. \tag{25.21}$$

In words, take the part of C_s where (x, y) are nonnegative, and look at it in logarithmic coordinates with base s. Then, as s goes to infinity, this converges to the corresponding tropical curve. We will leave the statement imprecise, by not explaining what notion of convergence appears here. The important point is that tropicalization can serve as an intermediate notion between the patchworking diagram and the actual algebraic curve, and thereby provides us with a better understanding of patchworking itself.

Example 25.5. *Take* (24.5)*, but replacing* (x, y) *by* $(-x, -y)$*, and setting* $s = t^{-1}$ *as before, which means*

$$\begin{aligned} C_s &= \big\{1 + x + y + s^{-2}x^2 - s^{-1}xy + s^{-2}y^2 + s^{-6}x^3 + s^{-4}x^2y + s^{-4}xy^2 + s^{-6}y^3 \\ &\qquad + s^{-12}x^4 - s^{-9}x^3y + s^{-8}x^2y^2 - s^{-9}xy^3 + s^{-12}y^4 = 0\big\}, \\ C_{\mathrm{trop}} &= \big\{ \max\{0, a, b, 2a-2, 2b-2, 3a-6, 2a+b-4, a+2b-4, 3b-6, \\ &\qquad 4a-12, 2a+2b-8, 4b-12\} = \max\{a+b-1, 3a+b-9, a+3b-9\}\big\}. \end{aligned} \tag{25.22}$$

We draw the patchworking diagram, which is a rotated version of the bottom left quadrant in (24.8)*, alongside* C_{trop} *and* $\mathrm{Log}_s(C_s)$ *(for* $s = 1000$*):*

(25.23)