26. Projective geometry

A century ago, there might have been an entire course in projective geometry! Here, two lectures is all we have time for.

- We define points and lines in the projective plane, and explain how they are related to standard planar geometry.
- We look at some properties of projective geometry, including a surprising duality between points and lines.

(26a) The projective plane. The projective plane, written here as $\mathbb{P}^2$, is made up of the following:

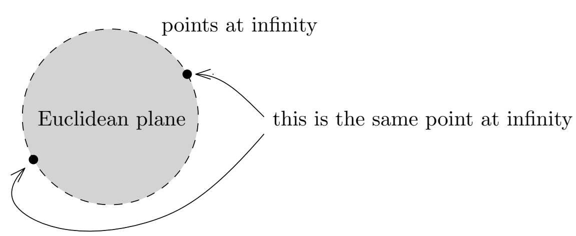
DEFINITION 26.1. *A projective point $p \in \mathbb{P}^2$ is a triple $[x : y : z]$, where $(x, y, z) \in \mathbb{R}^3$ cannot all be zero, and with the convention that $[x : y : z]$ and $[tx : ty : tz]$ are the same point, for all $t \neq 0$. The $[x : y : z]$ are called homogeneous coordinates, and we use the $[::]$ notation to remind ourselves that this denotes a projective point.*

One can think of projective points as lines through the origin in $\mathbb{R}^3$. The point $[x : y : z]$ corresponds to the line in $\mathbb{R}^3$ consisting of all multiples of the vector (x, y, z). That explains why we can't have $[0 : 0 : 0]$ (not a line in space), and also why $[x : y : z]$ and $[tx : ty : tz]$ are the same point (they give the same line in space). The relation with ordinary plane geometry is done as follows:

- Each point (x, y) in $\mathbb{R}^2$ becomes a projective point $[x : y : 1] \in \mathbb{P}^2$. This gets you all those projective points whose homogeneous z-coordinate is nonzero, because $[x : y : z] = [x/z : y/z : 1]$ for $z \neq 0$.
- The remaining projective points $p = [x : y : 0]$, which do not belong to the Euclidean plane, are called *points at infinity*. A point at infinity corresponds to a line through the origin in $\mathbb{R}^2$. More intuitively, one can think of points at infinity as corresponding to directions in $\mathbb{R}^2$, but where a direction and its opposite give the same point at infinity.

With that in mind, one can draw the projective plane qualitatively as follows:

(26.1)



Even though the idea of "points at infinity" is helpful for visualizing things, within projective geometry itself, this is not a natural distinction: any projective point is as good as any other one.

DEFINITION 26.2. *Take $(a, b, c) \neq (0, 0, 0)$. A projective line $L \subset \mathbb{P}^2$ consists of all $p = [x : y : z]$ which solve the equation $ax + by + cz = 0$.*

We can think of projective points as lines in $\mathbb{R}^3$, and correspondingly of projective lines as planes through the origin in $\mathbb{R}^3$, whic consist of all solutions (x, y, z) of $ax + by + cz = 0$. Then, a projective point p lies on a projective line L iff the line in $\mathbb{R}^3$ corresponding to p is contained in the plane corresponding to L. Again, there's a relation with the standard geometry of $\mathbb{R}^2$, with one exception:

- Suppose that $(a, b) \neq (0, 0)$. In that case, the associated projective line consists of points $[x : y : 1]$ which satisfy $ax + by + c = 0$, which is an ordinary line in $\mathbb{R}^2$; together with one point at infinity, which is the unique solution $[x : y : 0]$ of $ax + by = 0$. We say that this projective line is the completion of $ax + by + c = 0$.
- if $(a, b) = (0, 0)$, we have the *line at infinity* $z = 0$, which consists of all points at infinity.

FACT 26.3. *Through any two (different) projective points, there is a exactly one projective line.*

In terms of $\mathbb{R}^3$, this means that if we take two different lines through the origin, then they lie on a uniquely determined common plane. While the same property holds in the Euclidean plane, the following statement wouldn't:

FACT 26.4. *Any two (different) projective lines intersect in exactly one projective point.*

In terms of $\mathbb{R}^3$, this means that if we take two different planes through the origin, their intersection is a line through the origin. From a viewpoint of standard plane geometry, this result looks like this:

- if you have two lines in $\mathbb{R}^2$ which are not parallel, their projective completions have different points at infinity. So the intersection of the completions still consists of one point in $\mathbb{R}^2$.
- If you have two parallel lines in $\mathbb{R}^2$, their projective completions have the same point at infinity, where they intersect.
- Finally, if we take the projective completion of a line in $\mathbb{R}^2$, that always intersects the line at infinity in one point.

(26b) Duality. There is a general principle, called *projective duality*, which allows us to switch the role of lines and points. The idea is very simple: we switch the line $L = \{ax + by + cz = 0\}$ with the point $p = [a : b : c]$, and vice versa. If one thinks of projective points and lines as linear subspaces of $\mathbb{R}^3$, then duality consists of passing to the orthogonal complement. With that in mind, we write $p^\perp$ for the dual (projective line) to the point p, and $L^\perp$ for the dual (point) of the projective line L. Duality has the following property:

$$p \text{ lies on } L \;\;\Leftrightarrow\;\; L^\perp \text{ lies on } p^\perp. \tag{26.2}$$

EXAMPLE 26.5. *If I take a point*

$$(a, b) \in \mathbb{R}^2, \;(a, b) \neq (0, 0), \tag{26.3}$$

that becomes $p = [a : b : 1]$ in the projective plane, which is the line in $\mathbb{R}^3$ consisting of multiples of $(a, b, 1)$. Its dual is $\{ax + by + z = 0\}$, which is the projective completion of

$$ax + by + 1 = 0. \tag{26.4}$$

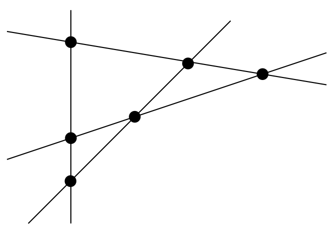
In contrast, if I take the origin $(0, 0)$ in $\mathbb{R}^2$, then the dual is the line at infinity $\{z = 0\}$.

A classical application of duality is to configurations of points and lines.

Definition 26.6. *Let c, γ, l, λ be integers, such that $c\lambda = l\gamma$. A $(c_\lambda l_\gamma)$* configuration *consists of c (different) points and l (different) lines in the projective plane, such that: each of the c points lies on exactly λ of the l lines; and each of the l lines contains exactly γ of the c points. (A configuration doesn't need to contain all the lines connecting its points, nor all the intersection points of its lines.)*

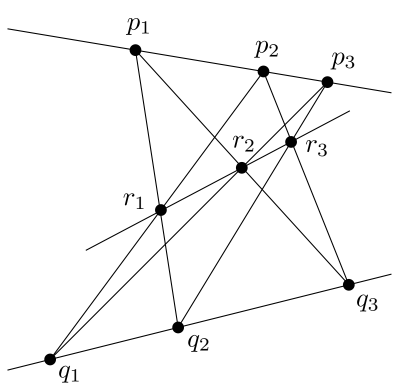
Example 26.7. *A complete quadrilateral consists of four lines, no three of which meet in a common point, and the 6 points in which two of those lines intersect. This is a $(6_2 4_3)$ configuration.*

(26.5)



Example 26.8. *Here's a $(9_3 9_3)$ configuration constructed starting from the points (p_1, p_2, p_3) which are collinear (lie on the same line), and three more points (q_4, q_5, q_6) which are also collinear (for a different line). We connect p_i with q_j for all $i \neq j$, adding 6 more lines. Pappus' theorem from geometry tells us that the intersection points r_1, r_2, r_3 are collinear, which yields the required ninth line in the configuration.*

(26.6)



The theory of configurations asks for what (p, λ, l, β) a configuration exists; and if there are ones with the same (p, λ, l, β) that are combinatorially or geometrically different from each other. This is best done in the projective plane, to avoid having to deal separately with the case of parallel lines.

PROPOSITION 26.9. *If we take a* $(c_\lambda l_\gamma)$ *configuration, and apply projective duality to all its points and lines, we get an* $(l_\gamma c_\lambda)$ *configuration.*

EXAMPLE 26.10. *The dual of a complete quadrilateral is a* $(4_3 6_2)$ *configuration: it consists of* 4 *points, no three of which are collinear, and all possible lines through two of those points.*

(26.7)