27. Algebraic curves in the projective plane

So far, all we've seen from projective geometry are points and lines.

- One can talk about projective algebraic curves, and this helps us understand the behaviour of ordinary algebraic curves (as they go out to infinity).
- There is a notion of nonsingularity for projective algebraic curves, and a corresponding version of Harnack's theorem. This situation is actually simpler than the case of $\mathbb{R}^2$, since we no longer have to distinguish between ovals and unbounded components.

(27a) Projective completion. A homogeneous polynomial of degree d, in variables (x, y, z), is an expression of the form

$$g(x, y, z) = \sum_{i+j+k=d} b_{ijk} x^i y^j z^k = 0, \tag{27.1}$$

where the coefficients (b_{ijk}) are not all zero. Note the $i+j+k = d$ condition here, every monomial has degree d. As a consequence,

$$g(tx, ty, tz) = t^d g(x, y, z), \tag{27.2}$$

In particular, whether $g(x, y, z)$ is zero or not depends only on the projective point $p = [x : y : z]$.

Definition 27.1. *Let $g(x, y, z)$ be a homogeneous polynomial of degree d. Then, the set $D \subset \mathbb{P}^2$ of those points $p = [x : y : z] \in \mathbb{P}^2$ where $g(x, y, z) = 0$ is called a* projective algebraic curve *of degree d.*

Suppose that $f(x, y)$ is a polynomial of degree d (not homogeneous), hence defines an algebraic curve $C \subset \mathbb{R}^2$. We can insert powers of z to make the polynomial homogeneous:

$$f(x, y) = \sum_{i+j\leq d} a_{ij} x^i y^j \implies g(x, y, z) = \sum_{i+j\leq d} a_{ij} x^i y^j z^{d-i-j}. \tag{27.3}$$

This defines a projective curve $D \subset \mathbb{P}^2$. As in the case of lines, we call this process *projective completion.* If we think of $\mathbb{P}^2$ as the union of $\mathbb{R}^2$ and the line at infinity, then D consists of C together with certain points at infinity. Those points at infinity are

$$D \setminus C = \{[x : y : 0] \ : \ h(x, y) = 0\}, \text{ where } h(x, y) = \sum_{i+j=d} a_{ij} x^i y^j. \tag{27.4}$$

In words, the points at infinity are defined using only those monomials in $f(x, y)$ which are of degree exactly d.

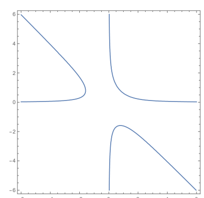
Example 27.2. *Let's look at what happens to certain kinds of conics in $\mathbb{R}^2$ under projective completion. We'll do this by example, but the behaviour only depends on the type of conic we're considering.*

type	*example*	*projective completion*	*equation for points at* ∞	*points at* ∞
ellipse	$x^2+2y^2=1$	$x^2+2y^2=z^2$	$x^2+2y^2=0$	*none*
parabola	$y=x^2$	$yz=x^2$	$0=x^2$	$[0:1:0]$
hyperbola	$xy=1$	$xy=z^2$	$xy=0$	$[1:0:0]$, $[0:1:0]$
parallel lines	$x(x-1)=0$	$x(x-z)=0$	$x^2=0$	$[0:1:0]$
intersecting lines	$xy=0$	$xy=0$	$xy=0$	$[1:0:0]$, $[0:1:0]$

The parabola has only one point at ∞, because its two ends go to ∞ in approximately the same direction. So do the two parallel lines, because if we go to ∞ in opposite directions, so end up at the same point of $\mathbb{P}^2$, by definition. One sees intuitively how this differs from the behaviour of the hyperbola and the intersecting lines.

Example 27.3. *The projective completion of $xy(x+y)=1$ is $xy(x+y)=z^3$. The points at infinity are solutions of $xy(x+y)=0$. There are three of them, $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$. This is clearly visible in the picture of the curve in $\mathbb{R}^2$, where there are three pairs of opposite directions in which the curve goes off to ∞:*

(27.5)



Example 27.4. *$x^4+y^2+1=0$ has no solutions in $\mathbb{R}^2$, but the projective completion is $x^4+y^2z^2+z^4=0$, which has the single solution $[x:y:z]=[0:1:0]$. The appearance of this point at infinity has no geometric motivation, but algebra rules here and we follow that.*

(27b) Nonsingular curves. Take a homogeneous equation $g(x,y,z)=0$. We say that a solution $[x:y:z]$ is singular if the gradient $(\nabla g)_{(x,y,z)}$ is zero; otherwise, it's nonsingular. The equation is called nonsingular if all its solutions are nonsingular points. If we take an ordinary plane curve $f(x,y)=0$ and projectively complete to $g(x,y,z)=0$, then the notion of nonsingularity for points $[x:y:1]$ agrees with the one we had defined before (by a computation which we omit). However, one still has to look at the points at infinity!

Example 27.5. *Let's return to our collection of conics.*

type	*projective completion*	*gradient*	*points at* ∞	*are the points at* ∞ *singular?*
parabola	$yz-x^2=0$	$(-2x,z,y)$	$[0:1:0]$	*nonsingular*
hyperbola	$xy-z^2=0$	$(y,x,-2z)$	$[1:0:0]$, $[0:1:0]$	*nonsingular*
parallel lines	$x(x-z)=0$	$(2x-z,0,-x)$	$[0:1:0]$	*singular*
intersecting lines	$xy=0$	$(y,x,0)$	$[1:0:0]$, $[0:1:0]$	*nonsingular*

The projective completions of the ellipse, parabola and hyperbola are nonsingular projective algebraic curves. In the second-to-last case, the point at infinity where the completions of the two *parallel lines cross becomes a singular point. One could say that our original curve is "singular at infinity".*

LEMMA 27.6. *Let $f(x, y)$ be a polynomial of degree d. If the projective completion of the associated curve has exactly d points at infinity, then those points must be nonsingular.*

To see that, let's suppose for simplicity that $[1 : 0 : 0]$ is not a point at infinity. With notation as in (27.3), look at this:

$$p(x) = g(x, 1, 0) = \sum_{i+j=d} a_{ij} x^i. \tag{27.6}$$

$p(x)$ is a polynomial of degree $\leq d$, and not identically equal to zero. By assumption, it has d roots x, which correspond to the points at infinity $[x : 1 : 0]$ of our completion. In that case, we necessarily have $p'(x) \neq 0$ at each root. Because

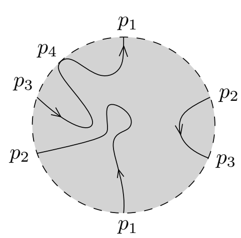
$$p'(x) = (\nabla g)_{(x,1,0)} \cdot (1, 0, 0), \tag{27.7}$$

this means that the gradient is nonzero at each such point.

THEOREM 27.7. *A nonsingular projective curve consists of a finite number of projective ovals (which one can think of as parametrized by embedded loops in $\mathbb{P}^2$).*

While making the notion of embedded loop in $\mathbb{P}^2$ rigorous would require quite a bit of work, the geometric intuition isn't that hard: it's a loop in the plane that can go off to infinity in some direction, and then either come back or re-emerge in the opposite direction. Here's such a loop, which with four points at infinity (three where it crosses the line at infinity, and one where it just touches).

(27.8)



EXAMPLE 27.8. *Take the projective completion of an ellipse, a parabola, or a hyperbola. Each has exactly one projective oval! In fact, from the point of view of projective geometry, they all look the same.*

EXAMPLE 27.9. *The completion of the curve from Example 27.3 is nonsingular (by Lemma 27.6), and has only one projective oval. One sees that just by following along as it goes through the points at infinity.*

Many of the results we have discussed for algebraic curves in $\mathbb{R}^2$ have better-behaved projective analogues. Here's Harnack's theorem:

THEOREM 27.10. *A nonsingular projective curve of degree d consists of at most $d(d-3)/2 + 2$ ovals.*