29. Betti numbers

We have mentioned shape complexes, but we didn't explain the meaning of the word "complex". What's going on is that any triangulation is an example of the much more general notion of *planar complex*. A planar complex is a collection points, edges (line segments), and triangles in the plane. In this lecture,

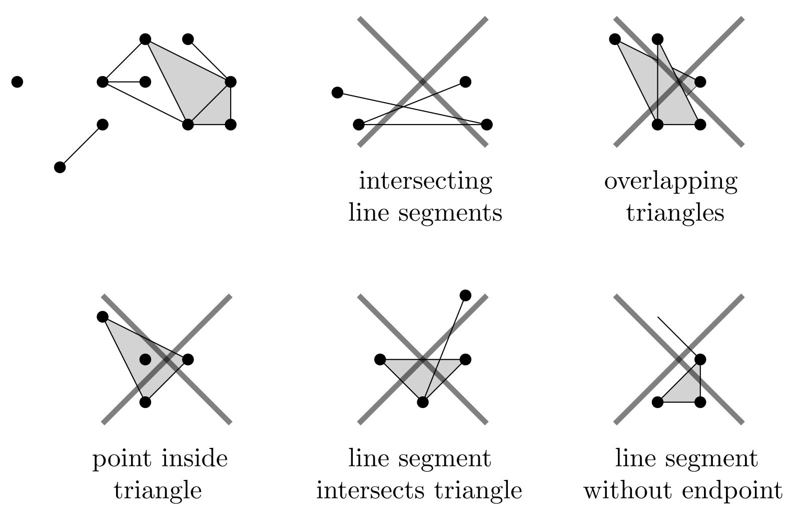
- We introduce planar complexes, and their Euler characteristic;
- we encode the combinatorial structure of such a complex in its boundary operators (which are matrices, so, get ready for some linear algebra);
- from those matrices, we extract some more interesting topological invariants, the Betti numbers.

(29a) Combinatorial data. A planar complex is: a finite collection of points; plus, a finite collection of edges (line segments); plus, a finite collection of triangles, all of it in the plane, and subject to a bunch of rules:

- If an edge is part of our complex, then both of its endpoints are part of the complex.
- If a triangle is part of our complex, then all three sides are edges that are also part of the complex.
- Otherwise, no overlaps, no intersections!

It may be easiest first to think of the case where there are only points and edges. Then, a planar complex is just a graph drawn in the plane (with straight edges that don't intersect). To make a general complex, we fill in some (could be none, all, or any subset of them) triangle-shaped regions created by that graph. Here's an example and some non-examples:

(29.1)



Definition 29.1. *Suppose that a planar complex K consists of n_0 points, n_1 edges, and n_2 triangles. Its* Euler characteristic $\chi = \chi(K)$ *is*

$$\chi = n_0 - n_1 + n_2. \tag{29.2}$$

Given that the Euler characteristic contains so little information about our complex (it only knows how many pieces of each dimension there are, not how they are arranged), it's surprising that it is of any importance at all!

(29b) Boundary operators. What if we were programmers, and wanted to encode the combinatorial structure of a complex? We could do it like this.

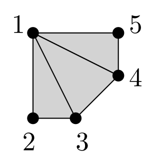
- Number the points by $\{1, \dots, n_0\}$.
- Every edge can be described by its pair (i, j) of endpoints, for $1 \leq i < j \leq n_0$. Record all the edges in our complex by pairs $(i_1, j_1), \dots, (i_{n_1}, j_{n_1})$.
- Every triangle can be described by its triple (p, q, r) of vertices, for $1 \leq p < q < r \leq n_0$. Record all the triangles in our complex by triples $(p_1, q_1, r_1), \dots, (p_{n_2}, q_{n_2}, r_{n_2})$.

Next, we turn the combinatorial data into a pair of matrices, the so-called boundary operators D_1 and D_2 of the complex.

D_1 is a matrix with n_0 rows and n_1 columns, which means that rows are labeled by points and columns are labeled by edges (the triangles are irrelevant for D_1). Each column vector contains one entry with -1 and one entry with 1, all other entries being zero. Namely, if the column corresponds to an edge (i, j), the i-th entry is -1 and the j-th entry is 1.

Example 29.2. *This is the complex obtained from a triangulation of a pentagon:*

(29.3)



It has $n_0 = 5$, $n_1 = 7$, $n_2 = 3$ (hence $\chi = 1$). The edges are

$$(1, 2),\ (1, 3),\ (1, 4),\ (1, 5),\ (2, 3),\ (3, 4),\ (4, 5). \tag{29.4}$$

Therefore,

$$D_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{29.5}$$

D_2 is a matrix with n_1 rows and n_2 columns, which means that rows are labeled by edges and columns are labeled by triangles. Each column vector contains two 1 entries and one -1 entry. Namely, if the column corresponds to a triangle (p, q, r), then the entries corresponding to the edges (p, q) and (q, r) are marked 1, and the entry corresponding to (p, r) is marked -1.

EXAMPLE 29.3. *For* (29.3)*, the triangles are*

$$(1,2,3),\ (1,3,4),\ (1,2,5). \tag{29.6}$$

The first triangle has edges $(1,2)$*,* $(2,3)$ *and* $(1,3)$*, which are numbers* 1*,* 5 *and* 2 *in the ordering from* (29.4)*. This determines where to put the nonzero entries in the first column of* D_2*. Taking this and the other two triangles into account, we get:*

$$D_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{29.7}$$

When writing down the matrices, we have implicitly chosen to order the edges and triangles in some way. We'll use lexicographic ordering, but that doesn't really matter. What matters is that when choosing the order in which the edges appear, you need to use the same one for D_1 and D_2 (as we've done in the examples above).

FACT 29.4. *The boundary operators always satisfy* $D_1 D_2 = 0$ *(the zero matrix).*

(29c) Betti numbers. Remember that the rank of a matrix A is the maximal number of linearly independent columns that you can find. It is also the maximal number of linearly independent rows, which means that a matrix and its transpose have the same rank:

$$\operatorname{rank}(A^t) = \operatorname{rank}(A). \tag{29.8}$$

The nullity of a matrix is the maximal number of linearly independent vectors w which solve $Aw = 0$, the linear system of equations determined by A. The rank-nullity theorem relates the two notions:

$$\text{if } A \text{ is a matrix with } n \text{ columns, } \operatorname{rank}(A) + \operatorname{nullity}(A) = n. \tag{29.9}$$

DEFINITION 29.5. *The Betti numbers* $b_0 = b_0(K)$*,* $b_1 = b_1(K)$*,* $b_2 = b_2(K)$*, are defined by*

$$\begin{aligned} b_0 &= n_0 - \operatorname{rank}(D_1), \\ b_1 &= n_1 - \operatorname{rank}(D_1) - \operatorname{rank}(D_2), \\ b_2 &= n_2 - \operatorname{rank}(D_2). \end{aligned} \tag{29.10}$$

Note that the alternating sum of the Betti numbers is the Euler characteristic:

$$b_0 - b_1 + b_2 = n_0 - n_1 + n_2 = \chi. \tag{29.11}$$

The Betti numbers are nonnegative integers. To see that, we use the linear algebra facts above:

$$\begin{aligned} b_0 &= n_0 - \operatorname{rank}(D_1^t) = \operatorname{nullity}(D_1^t), \\ b_1 &= \operatorname{nullity}(D_1) - \operatorname{rank}(D_2), \\ b_2 &= \operatorname{nullity}(D_2). \end{aligned} \tag{29.12}$$

From that, it's clear that $b_0 \geq 0$ and $b_2 \geq 0$. What about b_1? Because $D_1 D_2 = 0$, every column of D_2 is a solution of $D_1 w = 0$, so there are at least as many linearly independent solutions as column vectors, which means that $\text{nullity}(D_1) \geq \text{rank}(D_2)$.

EXAMPLE 29.6. *In* (29.5)*, the last four rows are clearly linearly independent. On the other hand, if we add up all the rows we get zero (something that's always true for* D_1*), so the first row is minus the sum of the others. It follows that* $\text{rank}(D_1) = 4$*. In* (29.7)*, the three columns are clearly linearly independent, so* $\text{rank}(D_2) = 3$*. Therefore, the Betti numbers are*

$$b_0 = 5 - 4 = 1, \; b_1 = 7 - 4 - 3 = 0, \; b_2 = 3 - 3 = 0. \tag{29.13}$$

It will take us a while to understand what Betti numbers mean, but here's a start:

THEOREM 29.7. b_0 *is the number of components (parts not connected to each other) of the complex.*

To understand that, think of what $D_1^t w = 0$ means. The vector w assigns to each vertex in our complex a real number. For each edge, the corresponding coefficient of $D_1^t w$ is the difference of the coefficients of w assigned to the endpoints of that edge. Therefore, $D_1^t w = 0$ says that whenever two vertices are connected by an edge, they carry the same number. So, a solution $D_1^t w = 0$ must assign the same value to all vertices in a given component, and there are no other constraints. In other words, such a solution is given by choosing a real number for each component.

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