30. Betti numbers (continued)

Picking up where we left off,

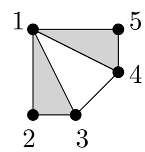
- we complete our discussion of Betti numbers of planar complexes.
- The definition of Betti numbers works for complexes in a more abstract sense, not drawn in the plane. This gives us examples with more interesting (and harder to understand) behaviour.

(30a) Betti numbers of planar complexes, revisited. Recall the definition of Betti numbers of a planar complex K, in terms of the ranks of boundary operators, and how we analyzed that using linear algebra:

$$\begin{aligned} b_0(K) &= n_0 - \mathrm{rank}(D_1) = \mathrm{nullity}(D_1^t), \\ b_1(K) &= n_1 - \mathrm{rank}(D_1) - \mathrm{rank}(D_2) = \mathrm{nullity}(D_1) - \mathrm{rank}(D_2), \\ b_2(K) &= n_2 - \mathrm{rank}(D_2) = \mathrm{nullity}(D_2). \end{aligned} \tag{30.1}$$

Example 30.1. *Take the example from the last lecture, but remove one triangle, so $n_0 = 5$, $n_1 = 7$, $n_2 = 2$:*

(30.2)



We have

$$D_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{30.3}$$

with $\mathrm{rank}(D_2) = 2$*, which means $b_2 = 0$. As we saw last time, b_0 is the number of components not connected to each other, so $b_0 = 1$. Finally, the Euler characteristic is*

$$\chi = b_0 - b_1 + b_2 = n_0 - n_1 - n_2 = 5 - 7 + 2 = 0 \tag{30.4}$$

from which we conclude that $b_1 = 1$ (one could of course also compute b_1 directly, using $\mathrm{rank}(D_1) = 4$*).*

Theorem 30.2. *For a planar complex K, we always have $b_2(K) = 0$.*

What we want to show is that $D_2 w = 0$ has only the trivial solution $w = 0$. When we spell it out, this is a linear system, with one variable w_T for each triangle, and one equation for each edge e. By definition of D_2, the equations have the form

$$\sum_{\substack{\text{triangles } T\\ \text{adjacent to } e}} \pm w_T = 0. \tag{30.5}$$

Hence, if T has an edge not shared by any other triangle, then $w_T = 0$; and if the edge e is shared by exactly two triangles T_1 and T_2, then $w_{T_1} = \pm w_{T_2}$. Starting with any triangle T, one can always pass through adjacent triangles (ones sharing an edge) until one reaches a triangle that has an "outside" edge, not shared with any other triangle. By going through all the equations, it follows that the coefficient of w_T the original triangle had to be zero.

Definition 30.3. *A* hole *of a planar complex K is a bounded component of the complement $\mathbb{R}^2 \setminus K$. Here, components means pieces not connected to each other; and bounded means that we exclude the infinite outside component.*

Theorem 30.4. *For a planar complex K, the Euler characteristic is*

$$\chi = (\textit{number of components of } K) - (\textit{number of holes of } K). \tag{30.6}$$

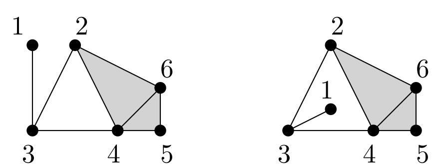
The main job here would be to prove the theorem about planar graphs (complexes without triangles). Once one has that, then filling in a triangle clearly raises χ by one and also destroys a hole, hence increases both sides of the equation by the same amount. We don't want to get too far into planar graphs, hence won't explain this further.

Corollary 30.5. *For every planar complex, b_1 is the number of holes.*

This follows from the previous results: by Theorem 30.2, $b_1 = \chi - b_0 - b_2 = \chi - b_0$. We also know (Theorem 29.7) that b_0 is the number of components; so by Theorem 30.4, b_1 must be the number of holes.

(30b) Abstract complexes. The definition of Betti numbers uses only data encoded into D_1 and D_2. Those data describe the adjacencies (how points, edges, and triangles fit together), but not how the complex lies in the plane. For instance, here are two complexes with the same adjacencies:

(30.7)



Maybe it would be better to say these are two pictures of the same "abstract" complex, but realized differently in the plane. In fact, Betti number can be defined in such an abstract situation, which is where they reach their full power.

Definition 30.6. *An* abstract complex *is given by combinatorial data, as follows:*

- *integers $n_0, n_1, n_2 \geq 0$.*
- *Pairs $(i_1, j_1), \ldots, (i_{n_1}, j_{n_1})$, where $1 \leq i_k < j_k \leq n_0$, and where no two pairs may be the same.*

- *Triples* $(p_1, q_1, r_1), \dots, (p_{n_2}, q_{n_2}, r_{n_2})$*, where* $1 \leq p_k < q_k < r_k \leq n_0$*, and where no two triples are the same. Moreover, whenever if a triple* (p, q, r) *appears, the pairs* (p, q)*,* (p, r)*,* (q, r) *must be on the previous list.*

We imagine these abstract points, edges and triangles glued together, all floating in your imagination (not in ordinary three-dimensional space: many abstract complexes can't be represented in three dimensions). The definition of Euler characteristic, boundary operator, and Betti numbers, go through as before. Also, the description of b_0 in terms of components still works for abstract complexes. In contrast, our description of b_1 makes no sense, since we don't have a complement of the complex. And finally, b_2 can be nonzero, as shown by the following example:

EXAMPLE 30.7. *Take a tetrahedron. It has* 4 *vertices and all possible edges and triangles,*

$$(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4) \quad \textit{and} \quad (1,2,3),\ (1,2,4),\ (1,3,4),\ (2,3,4). \tag{30.8}$$

The Euler characteristic is $\chi = n_0 - n_1 + n_2 = 4 - 6 + 4 = 2$*. The boundary operators are*

$$D_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{30.9}$$

As usual, the rows of D_1 *add up to zero, giving one linear relation; and the first three are linearly independent, so* $\mathrm{rank}(D_1) = 3$*. The alternating sum of the columns of* D_2 *(first minus second plus third minus fourth) is zero; and the first three columns are linearly independent, so* $\mathrm{rank}(D_2) = 3$*. We get*

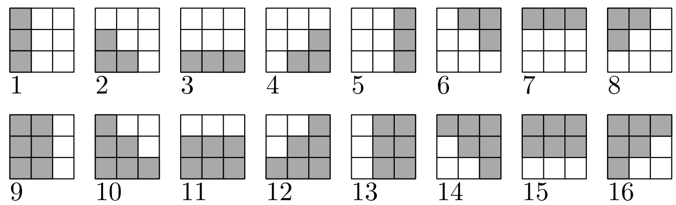
$$b_0 = 4 - 3 = 1, \quad b_1 = 6 - 3 - 3 = 0, \quad b_2 = 4 - 3 = 1. \tag{30.10}$$

(30c) The topology of data, revisited. Suppose we have points numbered $1, \dots, n_0$, and some notion of distance $\mathrm{dist}(i, j)$ between two points. They don't need to lie in the plane: they could be in a higher-dimensional space, or even in some more abstract context, and you can define distance in whichever way you want, subject to some commonsense constraints.

Fix some scale $\sigma > 0$. To our points, add edges (i, j) for each $i < j$ such that $\mathrm{dist}(i, j) < \delta$. In the same way, add a triangle (p, q, r) for each $p < q < r$ such that all three points are at distance $< \sigma$ from each other. The outcome is an abstract complex called the *Vietoris-Rips complex* of our point set, at scale σ. The Betti number b_0 can be thought of as a simple measure of clustering: we group our points so that any two with distance $< \sigma$ lie in the same group, and then b_0 is the number of groups. b_1 is a more interesting notion: it expresses insights about the structure of our point set which are not immediately obvious. (Finally, b_2 does not contain any meaningful information, due to the limitations of our setup).

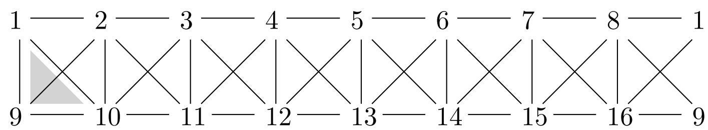
EXAMPLE 30.8. *Take these sixteen 3×3 pixel images:*

(30.11)



These will be the points of our abstract complex! We define the distance between two images to be the number of pixels whose differ. For instance, the distance between the 1st and 13th image is the maximal possible value, 9. Take $\sigma = 3.5$, and draw the edges of the Vietoris-Rips complex:

(30.12)



When drawing this in the plane, you'll see spurious intersections between the edges, which you are supposed to ignore. Moreover, to make the picture less messy, we have drawn two copies of the 1 and 9 points, but those should be thought of as being the same. To form the Vietoris-Rips complex, we should fill triangles wherever we can (we have indicated one triangle in the picture; when drawn in the plane, the triangles will overlap, hence we won't try drawing all of them). Altogether, we have $n_0 = 16$, $n_1 = 40$, $n_2 = 32$. Clearly, the whole complex is connected, so $b_0 = 1$. Rather than writing down D_2, I will give you a free piece of information, namely that $b_2 = 8$. Because of the Euler characteristic, this means that

$$b_1 = 1. \tag{30.13}$$

Intuitively, this is in agreement with imagining (30.11) as (two slightly different versions, one in each row, of) an image being rotated once, creating a "loop".

MIT OpenCourseWare
https://ocw.mit.edu

18.900 Geometry and Topology in the Plane
Spring 2023

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.