31. Surfaces

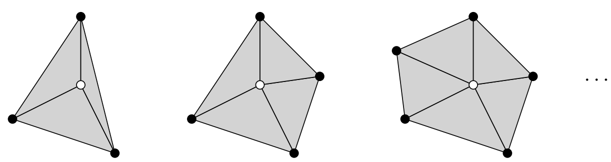
We focus on a very special class of abstract complexes, namely combinatorial surfaces.

- Orientability is a key distinction between such surfaces. We will look at examples of orientable and non-orientable surfaces.
- We study the implications of orientability for Betti numbers.

(31a) Combinatorial surfaces. A *combinatorial surface* is an abstract complex, such that:

- Every edge appears on the boundary of exactly two triangles.
- Take any vertex, and look at the edges and triangles that have our chosen vertex lying on them. Then, the outcome looks like one of these (with the chosen vertex marked in white):

(31.1)



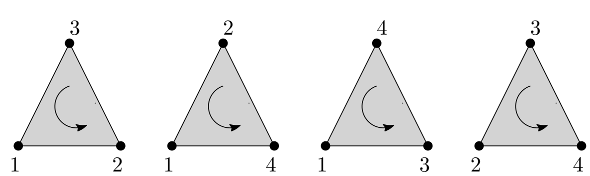
This means that the adjacent triangles together look like a convex polygon, triangulated by connecting an interior point to all its vertices.

Definition 31.1. *Suppose that for each triangle, we choose one of the two possible ways of going around its boundary, subject to this constraint: for any given edge, the choices for the two adjacent triangles yield opposite ways of going along that edge. This is called an* orientation *of the surface. It's not always possible: a surface which allows it is called orientable.*

(31b) Examples. We already saw one abstract complex, namely the tetrahedron, which is actually a combinatorial surface. The same holds for the octahedron, icosahedron, and other (less symmetric) convex polyhedra with triangular faces: they are all surfaces, and can be thought of as combinatorial versions of the two-dimensional sphere.

Example 31.2. *The tetrahedron is orientable (and the same holds for the other combinatorial spheres). This is easiest to see by drawing the triangles one-by-one, and then picking a way to go around the boundary of each, so that the edge conditions are satisfied:*

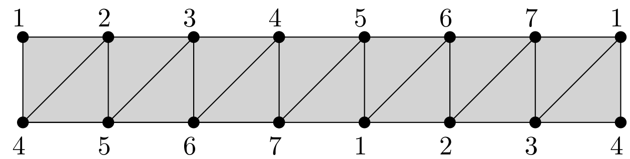
(31.2)



For instance, in the first triangle, we go around the boundary by moving from 1 *to* 2*; whereas in the second triangle, we move from* 2 *to* 1*.*

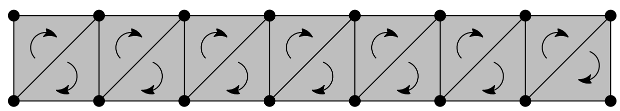
Example 31.3. *The following picture,*

(31.3)



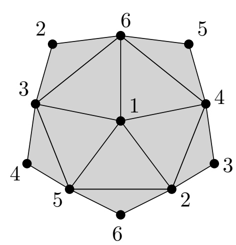
properly understood, is a combinatorial surface (a torus*). To represent it in the plane, we have drawn several copies of the vertices, and that also holds for the edges: the* $(1,4)$ *edges on the left and right side are the same. If we cut out the picture and glue those two sides together, we get a ring (annulus), but note that the top and bottom sides should also be thought of as being glued together. We have* $n_0 = 7$, $n_1 = 21$, $n_2 = 14$, *hence* $\chi = 0$. *The torus is orientable (remember, you have to check that the orientation condition also holds at the edges that have been drawn twice):*

(31.4)



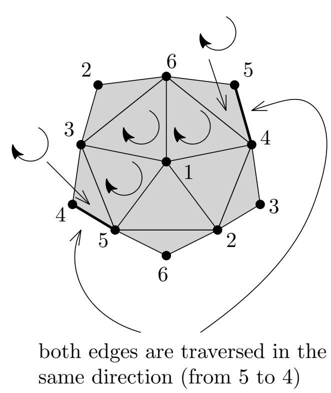
Example 31.4. *Here is another picture of the same kind as the previous one. Note that on the boundary of our picture, every point and every edge is identified with its counterpart on the opposite side. One can think of it as the top half of an icosahedron, where the boundary is glued to itself with a* 180 *degree twist. In fact, it is a combinatorial version of the projective plane.*

(31.5)



We have $n_0 = 6$, $n_1 = 15$, $n_2 = 10$, *which means that* $\chi = 1$. *The projective plane is non-orientable. To see this, it's enough to start with one triangle and gradually try to extend orientations to the neighbouring ones. The outcome is a contradiction:*

(31.6)



(31c) Orientability and its consequences. One reason why orientability is important is that it has significant implications for the Betti numbers.

PROPOSITION 31.5. *Take a combinatorial surface which is connected (meaning that it's not divided into several mutually disconnected parts; equivalently, $b_0 = 1$). Then $b_2 = 1$ if the surface is orientable, and $b_2 = 0$ otherwise.*

EXAMPLE 31.6. *For the torus from Example 31.3, we now know that $b_2 = 1$, and of course $b_0 = 1$. By the Euler characteristic computation, we must have $b_1 = 2$.*

EXAMPLE 31.7. *For the projective plane from Example 31.4, we now know that $b_2 = 0$, and also $b_0 = 1$, hence $b_1 = 0$.*

The proof of the Proposition is based on $b_2 = \text{nullity}(D_2)$, and an argument similar to that which showed $b_2 = 0$ for planar complexes. Suppose that we have oriented our surface, and take one of the triangles (p, q, r), for $1 \leq p < q < r \leq n_0$. Assign to our triangle a number ± 1, like this. If the orientation tells us to go around the triangle from the p-th point to the q-th point to the r-th point, we take $+1$; and if the opposite is true, take -1. The condition on the orientation means that this collection of numbers is a solution to $D_2 w = 0$, so its existence proves that $b_2 > 0$. The rest of the argument (showing that any other solution is a multiple of this one; and the converse direction, namely that existence of a solution implies orientability) is similar, and we won't go through it here.

PROPOSITION 31.8. *The Euler characteristic of an orientable surface is always even.*

Taking those two Propositions together, we also see this:

COROLLARY 31.9. *For an orientable surface, b_1 is even.*

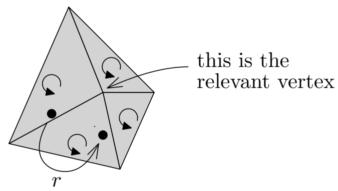
There is an elementary combinatorial proof of Proposition 31.8. Like many elegant arguments, it is also mystifying in a what-did-we-just-do way. Moreover, it relies on the notion of *sign of a permutation*, which we've not used before; so, you have my permission to skip it if you want!

Take a surface which has been oriented. Define a *side* to be an edge together with the choice of one of the two adjacent triangles. Let's call the set of sides Σ (it is of size $2n_1$). We look at three ways of permuting the sides:

- The opposite map $o : \Sigma \to \Sigma$, which keeps the edge but passes from one adjacent triangle to the other. In other words, for each edge, it swaps out the two possible sides. As a consequence, $\text{sign}(o) = (-1)^{n_1}$.
- The successor map $s : \Sigma \to \Sigma$ uses the orientation, to go from the given side to the next one for the same triangle. Clearly, if we do it three times, we get back to the original side, meaning that s^3 is the trivial (identity) permutation. This shows that $\text{sign}(s)^3 = \text{sign}(s^3) = 1$, and therefore $\text{sign}(s) = 1$.
- The rotator map $r : \Sigma \to \Sigma$ is a little more complicated. Given a side, move forward (using the orientation of the triangle) along the edge until one hits a vertex. We then

pass to the next triangle adjacent to that vertex, again using the orientation:

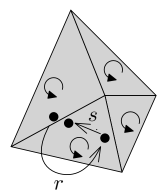
(31.7)



Here, we've indicated a side just by drawing a dot in the triangle, lying near the desired edge. sign(r) is the number of vertices of our surface which have even valence (have an even number of edges adjacent to them); to see that, one needs to look at how each side cycles if we repeat r.

One observes (proof-by-picture) that these three permutations are related, one being the composition of the other two:

(31.8) $$o = s \circ r$$



As a consequence, sign(o) = sign(s)sign(r) = sign(r). In words, the number of edges is congruent mod 2 to the number of even-valence vertices. At this point, we need two more easy combinatorial facts (the first is true for any graph, and the second for any surface):

- the number of vertices of odd valence is even;
- the number of triangles is even.

The previous argument, and the first fact, combine to show that the number of edges is congruent mod 2 to the number of all vertices. Together with the second fact, we see that the number of edges is congruent mod 2 to the number of vertices plus the number of triangles. Which is exactly what Proposition 31.8 said!

(31d) Summary. Since we have talked about Betti numbers in various degrees of generality, it makes sense to summarize we know about their behaviour and geometric meaning (for surfaces we have assumed connectedness, to make the statements simpler; of course, in general a surface doesn't have to be connected).

	planar complex	abstract complex	connected orientable surface	connected non-orientable surface
b_0	components	components	1	1
b_1	holes	?	even	?
b_2	0	?	1	0