32. Combinatorial loops

In an abstract complex, a combinatorial loop is a path that moves along edges and ends where it started.

- We introduce the notion of homotopy (a combinatorial version of deformation) between two such loops.
- The main question is: how can we decide if two loops are homotopic to each other?

We will look at a number of examples, and answer our question by methods that look a bit improvised, even though they will remind you of our previous topics of polygonal/smooth loops and their winding numbers.

(32a) The definitions. Take an abstract complex K with n_0 points (vertices). A *combinatorial loop* of length $m \geq 0$ is a collection $l = (a_0, a_1, \dots, a_m)$ of points in K, meaning that $1 \leq a_k \leq n_0$, such that:

- for all k, we have $a_{k-1} \neq a_k$, and there is an edge connecting a_{k-1} to a_k (in other words, if $a_{k-1} < a_k$, then (a_{k-1}, a_k) is in the list of edges for K; and if $a_{k-1} > a_k$, then (a_k, a_{k-1}) is in that list).
- $a_m = a_0$.

In words, the loop moves from vertex to vertex along edges, and eventually returns to its starting point. The simplest loops are the constant ones, $m = 0$, which just consist of one vertex $l = (a)$. The next simplest ones are zigzags $(a_0, a_1, a_2 = a_0)$, which have $m = 2$.

What makes combinatorial loops interesting is the notion of homotopy. Two loops are homotopic if they are related by a sequence of the following moves:

- Changing the starting point: this means passing from $(a_0, \dots, a_m)$ to $(a_1, a_2, \dots, a_m, a_1)$, or vice versa to $(a_{m-1}, a_0, a_1, \dots, a_{m-1})$. This makes sense because $a_m = a_0$.
- Removing or inserting a zigzag: if $(...a, b, a...)$ occurs in our loop, then we can replace that by $(\dots a \dots)$, which shortens it by 2. The reverse move, which makes a loop longer by 2, is also allowed.
- Moving across a triangle: if our loop has $(...a, b, c...)$ and $\{a, b, c\}$ are the three vertices of a triangle (in any order), then we can delete b, shortening the loop by 1. The reverse move, which makes a loop longer by 1, is also allowed.

Lemma 32.1. *For a constant loop, the homotopy class depends only on the component of K in which it lies.*

Suppose there's an edge connecting the a-th and b-th vertex. Then the constant loops (a) and (b) are homotopic:

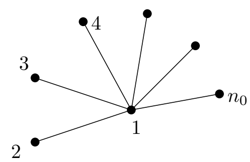
$$(a) \xrightarrow{\text{insert zigzag}} (a, b, a) \xrightarrow{\text{change starting point}} (b, a, b) \xrightarrow{\text{remove zigzag}} (b). \tag{32.1}$$

By saying that K has only one component, we mean that you can move from any vertex to any other along edges. By repeating the argument above, we then see that any two vertices give rise to homotopic constant paths.

(32b) Examples. The simpler half of the problem is the constructive one: to show that a given loop is homotopic to a constant, or that two loops are homotopic to each other, all one needs to do is find a suitable sequence of moves.

EXAMPLE 32.2. *Take this star-shaped graph (a complex without triangles):*

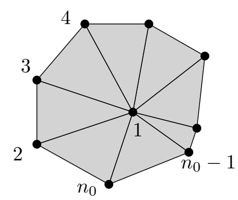
(32.2)



Any non-constant loop must pass through the 1 *vertex. After rotating the starting point, we can assume that it starts and ends at* 1*. Then, it necessarily consists of zigzag pieces* $(\dots 1, a, 1, \dots)$ *with* $1 < a \leq n_0$*. Each such piece can be cancelled. It follows that every loop is homotopic to a constant loop.*

EXAMPLE 32.3. *Take this complex (with* $n_0 \geq 4$ *points, forming an* $(n_0 - 1)$*-gon with one point at the center):*

(32.3)

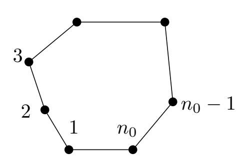


Suppose we have a loop $(...a, b...)$*, where* $a, b \geq 2$*. By moving across a triangle, one can always replace that by* $(...a, 1, b...)$*. Having done that in all places, we have a loop that bounces back-and-forth between* 1 *and other vertices, which then can be shortened to a constant. Hence, all loops are homotopic to constant ones.*

We now turn to the harder theoretical part of the problem, where one is trying to prove that a given loop is not homotopic to a constant, or that two loops are not homotopic. For that, one has to find some sort of obstacle that will prevent one loop from turning into the other. Here's a toy model:

EXAMPLE 32.4. *Take this graph (*$n_0 \geq 3$ *points connected to each other in a circular way, with no triangles):*

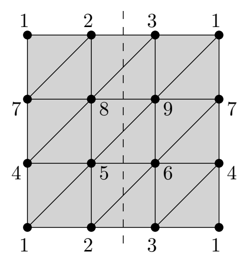
(32.4)



We have a strong intuitive feeling that the loop $(1, 2, 3, \dots, n_0, 1)$ *is not homotopic to a constant loop, since it "goes once around". To make it rigorous, we introduce a combinatorial analogue* *of the notion of winding number. Namely, we associate to every loop l an integer $I(l)$, like this: whenever $(...1, 2...)$ occurs in our loop, count that as $+1$; whenever $(...2, 1...)$ occurs, count it as -1; and add up those numbers. This integer is unchanged under homotopies (there are no triangles, so we don't have to look at that move). Now, the loop we started with had value $w(l) = 1$, whereas the constant loop has $w(l) = 0$, so they can't be homotopic.*

(32c) The torus. Take this surface, which is a combinatorial version of a torus:

(32.5)



We can associate to a loop a winding number $w(l)$, by counting how often it crosses the dashed cut drawn above, with signs: $+1$ for a left-to-right crossing, and -1 for a right-to-left one. This means that we count occurrences of the following patterns:

$$\begin{aligned} &(...2,3...),\ (...8,9...),\ (...5,6...),\ (...8,3...),\ (...5,9...),\ (...2,6...) \text{ count as } +1, \\ &(...3,2...),\ (...9,8...),\ (...6,5...),\ (...3,8...),\ (...9,5...),\ (...6,2...) \text{ count as } -1. \end{aligned} \tag{32.6}$$

The sum of all those signs is unchanged under homotopies! One can show that by checking it for every possible occurrence of move-over-a-triangle.

EXAMPLE 32.5. *The loop $l = (1, 2, 3, 1)$ has $w_{\rightarrow}(l) = 1$. Hence, it's not homotopic to a constant, but it's also not homotopic to $l = (1, 2, 3, 1, 2, 3, 1)$, which has $w_{\rightarrow}(l) = 2$.*

EXAMPLE 32.6. *The loop $l = (1, 4, 6, 9, 8, 2, 1)$ has $w_{\rightarrow}(l) = -1$ and $w_{\uparrow}(l) = 1$.*

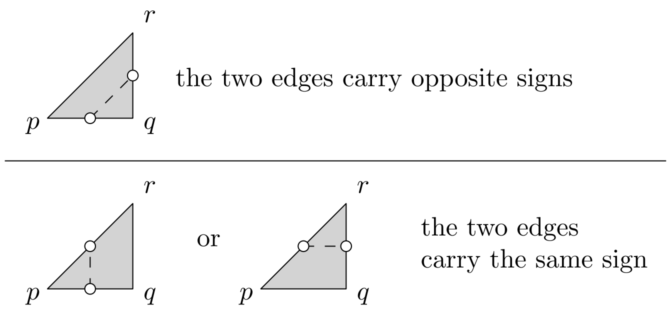
There is also an entirely different vertical winding number, which would be obtained by counting the crossings with a left-to-right cut.

(32d) General cuts. A cut c in an abstract complex is defined by picking a set of edges (which we picture by drawing a dot in the center of each edge) and a sign -1 or $+1$ on each of those edges, obeying the following rules:

- For every triangle in the complex, either none or two of its boundary edges belong to the cut (we then visualize that by drawing a dotted line connecting the two).

- The signs (where the triangle has vertices p, q, r with $p < q < r$) must satisfy:

(32.7)



Each cut gives us a winding number $w_c(l)$ for loops l. Roughly speaking, whenever an edge in the cut appears as part of our loop ("the loop crosses the cut"), we get a ± 1 contribution, and the sum of those contributions is $w_c(l)$. The rule for the signs is: if an edge (i, j), obviously with $i < j$, is part of the cut, and our loop goes from i to j, use sign for that edge; and if the loop goes from j to i, use the opposite sign (as you can see, sign issues are rather tricky). The outcome is an integer which is unchanged under homotopies.

Example 32.7. *Here's our previous torus example, with the signs added:*

(32.8)