33. Combinatorial winding numbers and the boundary operators

In the previous lecture, we introduced cuts and the resulting winding numbers. We now generalize that notion a bit, and relate it to boundary operators.

- To each loop one can associate a vector, which counts how many times each edge in the complex appears in it (with signs).
- By taking suitable scalar products, one can define general combinatorial winding numbers, which are homotopy invariants. This construction will also explain the geometric meaning of the first Betti number of a complex.

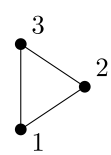
(33a) From loops to vectors. Let K be an abstract complex. Let's introduce some notation. If (i,j) is an edge, which by definition means $1 \leq i < j \leq n_0$, we write $e_{(i,j)}$ for the corresponding unit (standard basis) vector in $\mathbb{R}^{n_1}$ (recall n_1 is the number of edges, and we usually order those edges lexicographically). We also find it convenient to write $e_{(j,i)} = -e_{(i,j)}$. To any combinatorial loop $l = (a_0, \dots, a_m)$ one can associate a vector

(33.1)
$$v_l = \sum_{i=1}^{m} e_{(a_{i-1},a_i)} \in \mathbb{R}^{n_1}.$$

In words: if $a_{i-1} < a_i$, we take the unit vector for the edge (a_{i-1}, a_i); if $a_{i-1} > a_i$, take minus the unit vector for the edge (a_i, a_{i-1}); and add up all those vectors to get v_l. A constant loop l gives rise to the vector $v_l = 0$, since there are no contributions at all. The same is true for loops $l = (a, b, a)$, since one gets two terms which exactly cancel each other. It is important to remember that v_l only sees which edges are part of l, not the order in which they occur. Therefore, it doesn't have all the information about the loop,

Example 33.1. *Take the graph*

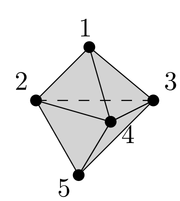
(33.2)



with edges $(1,2)$, $(1,3)$, $(2,3)$. *The loop* $l = (1,2,3,1)$ *consists of the edges* $(1,2)$, $(2,3)$, *and the reverse of* $(1,3)$. *Therefore,* $v_l = (1,-1,1)$.

Example 33.2. *Take two tetrahedra, stick them together along one triangle, and then forget that triangle. The outcome is this surface* ($n_0 = 5$, $n_1 = 9$, *and* $n_2 = 6$ *including two triangles that are "hidden" at the back the picture):*

(33.3)



In the loop $l = (2, 3, 4, 2, 3, 4, 2)$*, each of the three edges* $(2, 3)$*,* $(2, 4)$*,* $(3, 4)$ *appears twice, but the edge* $(2, 4)$ *appears in reverse order. Therefore,*

$$v_l = 2e_{(2,3)} + 2e_{(3,4)} + 2e_{(4,2)} = 2e_{(2,3)} + 2e_{(3,4)} - 2e_{(2,4)} = (0, 0, 0, 2, -2, 0, 2, 0, 0) \in \mathbb{R}^{n_1}. \tag{33.4}$$

In the rightmost expression, we have followed our usual convention of listing the edges in lexicographic order: $(1, 2)$*,* $(1, 3)$*,* $(1, 4)$*,* $(2, 3)$*,* $(2, 4)$*,* $(2, 5)$*,* $(3, 4)$*,* $(3, 5)$*,* $(4, 5)$*.*

Lemma 33.3. *The vector* v_l *always satisfies* $D_1 v_l = 0$*.*

For that, it's enough to remember that by definition of boundary operators, $De_{(i,j)} \in \mathbb{R}^{n_0}$ is the j-th unit vector minus the i-th unit vector. Basically, this the difference between the endpoints of the edge (i, j). The terms in $D_1 v_l$ coming from subsequent edges will partly cancel, since each edge ends where the following one starts; and because we have a loop that comes back to its starting point, they will finally cancel altogether.

Theorem 33.4. *Suppose that* l_0 *and* l_1 *are homotopic. Then* $v_{l_0} - v_{l_1} = D_2 x$ *for some* $x \in \mathbb{R}^{n_2}$*.*

To prove that, we have to investigate what happens to v_l under the moves that define the notion of homotopy. If we change the starting point, v_l doesn't change at all, since we still have the same edges, just in different order. And if insert or delete a zigzag $(\dots, a, b, a, \dots)$, we add or remove two contributions to v_l, but those contributions are the same basis vector with opposite signs, so again v_l remains the same.

The interesting part is passing over a triangle: a single move passes from $l_0 = (\dots, a, b, c, \dots)$ to $l_1 = (\dots, a, c, \dots)$. This means that

$$v_{l_0} - v_{l_1} = e_{(a,c)} - e_{(a,b)} - e_{(b,c)}. \tag{33.5}$$

The notation here hides some sign issues, but irrespectively, the right hand is, up to an overall sign, D_2 of the unit vector in $\mathbb{R}^{n_2}$ associated to the triangle with vertices $\{a, b, c\}$. A general homotopy involves several such moves, but one can add the resulting vectors in $\mathbb{R}^{n_2}$ to get the desired x.

(33b) Winding numbers. We want to turn the vectors v_l into a practical tool for distinguishing non-homotopic loops. For that purpose, it's important to remember the fact that $D_2 D_1 = 0$.

Corollary 33.5. *Fix some* $c \in \mathbb{R}^{n_1}$ *such that* $D_2^t c = 0$*. Then, the number* $c \cdot v_l \in \mathbb{R}$ *is a homotopy invariant, which means it remains the same if (keeping* c *the same, of course) we change* l *to a homotopic loop.*

We call this the combinatorial winding number of l with respect to c, and write it as

$$\mathrm{wind}_c(l) = c \cdot v_l. \tag{33.6}$$

The proof is really easy. Suppose that l_0 and l_1 are homotopic. Then

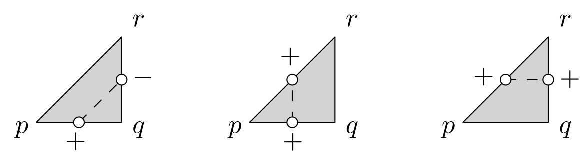
$$v_{l_1} - v_{l_0} = D_2 x \implies I_c(l_1) - I_c(l_0) = c \cdot (v_{l_1} - v_{l_0}) = c \cdot D_2 x = D_2^t c \cdot x = 0. \tag{33.7}$$

Concretely, a vector c has one coefficient for every edge, and $D_2^t c = 0$ consists of one equation for every triangle. If (p, q, r) is a triangle, which always means $p < q < r$, then the equation is:

(33.8) the (p, q)-coefficient of c plus the (q, r)-coefficient of c equals the (p, r)-coefficient of c.

The cuts used in the previous lecture were actually specific choices of vectors c: whenever an edge (i, j), $i < j$, occurs in the cut with sign ± 1, we take $\pm e_{(i,j)}$, and add up those vectors. The outcome satisfies (33.8), as one can check by looking at this picture:

(33.9)



$e_{(p,q)} - e_{(q,r)}$ $\quad$ $e_{(p,q)} + e_{(p,r)}$ $\quad$ $e_{(p,r)} + e_{(q,r)}$

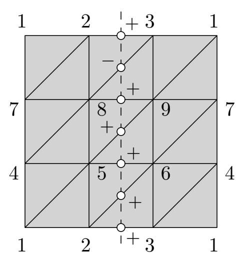
as standing for vectors (up to overall ± 1 signs)

$$c = \pm(e_{(p,q)} - e_{(q,r)}), \;\; c = \pm(e_{(p,q)} + e_{(p,r)}), \;\; c = \pm(e_{(q,r)} - e_{(p,r)}). \tag{33.10}$$

Of course, a general solution of $D_2^t c = 0$ doesn't correspond to a cut, and one could skip the geometric intuition and just look for such solutions directly, by solving that system of equations like any linear algebra problem.

Example 33.6. *The cut in the torus we drew last time,*

(33.11)



corresponds to the vector

$$c = e_{(2,3)} + e_{(5,6)} + e_{(8,9)} - e_{(3,8)} + e_{(5,9)} + e_{(2,6)}. \tag{33.12}$$

(33c) Theory aspects. Remember that $D_1 D_2 = 0$. For the transposed matrices, we have

$$D_2^t D_1^t = (D_1 D_2)^t = 0. \tag{33.13}$$

In principle, this seems to provide an easy way to find solutions of $D_2^t c = 0$: any vector $c = D_1^t b$ will do. However, these are useless for our purpose:

Lemma 33.7. *If* $c = D_1^t b$, *then* $\mathrm{wind}_c(l) = 0$ *for all loops* l.

The proof is a simple matrix computation: since $D_1 v_l = 0$,

$$\mathrm{wind}_{D_1^t b}(l) = (D_1^t b) \cdot v_l = b \cdot (D_1 b) = 0. \tag{33.14}$$

What is the overall implication? We have seen that any $c \in \mathbb{R}^{n_1}$ with $D_2^t c = 0$ gives rise to a combinatorial winding number wind_c. The number of linearly independent such c is

$$\mathrm{nullity}(D_2^t) = n_1 - \mathrm{rank}(D_2^t) = n_1 - \mathrm{rank}(D_2). \tag{33.15}$$

However, we now that some of those combinatorial winding numbers are just zero. The number of linearly independent c which are useless in this way is

$$\mathrm{rank}(D_1^t) = \mathrm{rank}(D_1). \tag{33.16}$$

Hence, the actually useful number is

$$\mathrm{nullity}(D_2^t) - \mathrm{rank}(D_1^t) = n_1 - \mathrm{rank}(D_2) - \mathrm{rank}(D_1) = b_1(K). \tag{33.17}$$

In words, this means that there are $b_1(K)$ essentially different ways of measuring "how a loop winds around K". Finally, this provides us with a geometric intuition for the first Betti numbers, even though it's one that requires quite a bit of background knowledge. The extreme case is $b_1(K) = 0$. In that case, the combinatorial winding numbers are all zero, which means they provide no information whatsoever about homotopy classes of loops. To understand homotopy better, one would then need to refine our tools (see the case of the projective plane from the end of the previous lecture; the mod 2 calculation used there hints at a whole new concept, that of mod 2 Betti numbers).