34. The hyperbolic plane

Hyperbolic geometry is one of the two non-Euclidean geometries. This means that it has the notions familiar to you from Euclidean geometry (points, lines, circles, distances, angles, areas), but most of them are interpreted in quite different ways.

- We define hyperbolic geodesics, which are the analogues of straight lines, and look at the beginnings of triangle geometry;
- We introduce the hyperbolic distance, and the associated notion of hyperbolic circle.

(34a) Points and lines. The hyperbolic plane is the upper half of the ordinary plane, minus the x-axis:

$$\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \;:\; y > 0\} = \{z = x + iy \in \mathbb{C} \;:\; \mathrm{im}(z) > 0\}. \tag{34.1}$$

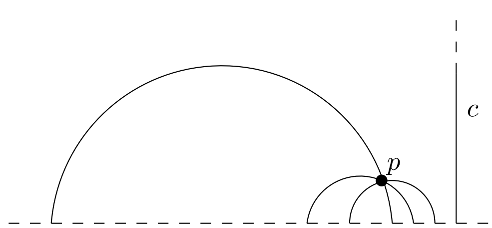
Complex coordinates turn out to be particularly useful in this context. The hyperbolic geometry notion of straight line has a special name:

Definition 34.1. *A* hyperbolic geodesic *in $\mathbb{H}$ is either a straight vertical half-line, or a half-circle centered on the horizontal axis.*

If we were living in the hyperbolic plane, Newtonian motion, light rays, and sound propagation, would happen along geodesics. To a person born with hyperbolic senses, all geodesics appear equivalent: the apparent distinction between circles and vertical lines is an artifact of the way we have represented the hyperbolic plane inside the ordinary plane. From elementary geometry, we can see that geodesics have some of the properties we expect from the notion of a straight line, but hold one surprise:

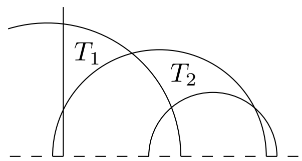
- Through any two points in $\mathbb{H}$ there is exactly one geodesic. (This implies that two different geodesics can intersect in at most one point.)
- If we fix a point in $\mathbb{H}$, there is exactly one geodesic through that point with any prescribed tangent line. (This implies that two different geodesics can never be tangent to each other.)
- Fix a geodesic c, and a point p not lying on c. Then there are infinitely many geodesics passing through p and which do not intersect c (this is unlike the case of Euclidean geometry, where the corresponding property characterizes the unique parallel line):

(34.2)



The last-mentioned fact shows that hyperbolic geometry is not Euclidean geometry written in some weird nonlinear coordinate system! Other notions are defined using geodesics. For instance, a triangle in hyperbolic geometry consists of three points joined by geodesic segments. Here are two examples:

(34.3)

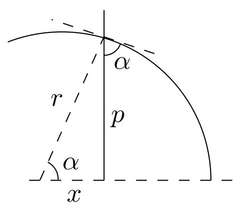


(34b) Angles. Hyperbolic geometry uses the same notion of angle as ordinary geometry. More precisely, if two geodesics intersect at a point, we take the tangent lines (in the standard sense) at that point, and measure their angle. As before, some familiar geometric properties hold, and some don't. What we are interested in is this:

Theorem 34.2. *In a hyperbolic triangle, the sum of the angles is always less than π.*

Let's first look at how a half-circle and a vertical line intersect:

(34.4)

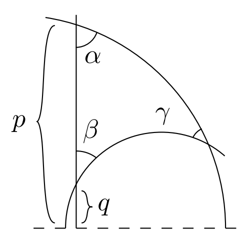


We are interested in the angle α of intersection, which is also one of the angles of the (Euclidean, not hyperbolic) right-angled triangle in the picture. Let x, p, r be the side-lengths (again in the standard Euclidean sense) of the triangle, so that $x = r\cos(\alpha)$ and $p = r\sin(\alpha)$. If we move the vertical line slightly, the angle changes like this:

$$\frac{d\alpha}{dx} = \left(\frac{dx}{d\alpha}\right)^{-1} = (-r\sin(\alpha))^{-1} = -1/p. \tag{34.5}$$

Look at a hyperbolic triangle where one of the three sides is a vertical line,

(34.6)



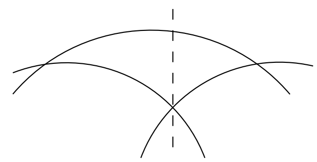
If I move the vertical line slightly to the right, the argument above says that $d\alpha/dx = -1/p$. A similar argument, using the complementary angle, shows that $d\beta/dx = 1/q$. Of course, γ is independent of x. The outcome is that

$$\frac{d}{dx}(\alpha + \beta + \gamma) = \frac{1}{q} - \frac{1}{p} > 0. \tag{34.7}$$

Qualitatively, as we move the vertical line to the right, the sum of angles increases. On the other hand, as we move the vertical line closer and closer to the intersection of the two circles, the hyperbolic triangle becomes tiny, and its behaviour is closer and closer to that of an ordinary straight-line triangle, so the sum of angles approaches π. We have just shown that in the original triangle (34.6), the sum of angles is always less than π!

This proves our desired theorem, in the special case where the left side of the hyperbolic triangle is a vertical line. Of course, by reflection, the same is true if the right side is a vertical line. Finally, for triangles bounded by three half-circles, we can obtain our result by decomposing them into two pieces which are triangles bounded by a vertical line, and then adding up the angles:

(34.8)



(34c) Distances. Remember that in the complex plane, $|z-w|$ is the ordinary distance between two points. In terms of the complex conjugate, one can write this as $|z-w|^2 = (z-w)(\bar{z}-\bar{w})$. The hyperbolic analogue is gruesome:

DEFINITION 34.3. *The hyperbolic distance between two points $z, w \in \mathbb{H}$ is*

$$\mathrm{dist}(z,w) = \ln\left(\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}\right). \tag{34.9}$$

A priori, it's not clear where this comes from, or that it makes any sense as a notion of distance. At least, since $|z-\bar{w}| = |w-\bar{z}|$, it is symmetric, meaning $\mathrm{dist}(z,w) = \mathrm{dist}(w,z)$.

EXAMPLE 34.4. *Take two points on the same vertical line, $w = x+iu$ and $z = x+iy$, with $u > y$. Then*

$$\mathrm{dist}(x+iu, x+iy) = \ln\left(\frac{(u+y)+(u-y)}{(u+y)-(u-y)}\right) = \ln(u/y) = \ln(u) - \ln(y). \tag{34.10}$$

We can make it more symmetric: for all $u \neq y$ (but still on the same vertical line),

$$\mathrm{dist}(x+iu, x+iy) = |\ln(u/y)| = |\ln(u) - \ln(y)|. \tag{34.11}$$

Unsurprisingly, hyperbolic geometry also uses hyperbolic trig functions (sinh, cosh, tanh; you may want to look up the definition and the shape of their graphs, to refresh your memory; in particular, the formula $\cosh(x)^2 - \sinh(x)^2 = 1$ is often useful). For us, their first appearance is in the following equivalent, and more convenient, formula for the distance:

$$\cosh(\mathrm{dist}(z,w)) - 1 = \frac{|z-w|^2}{2\,\mathrm{im}(z)\,\mathrm{im}(w)}. \tag{34.12}$$

Since cosh is defined in terms of exponentials, its inverse function can be written in terms of logarithms, more precisely it is $\pm\ln(x+\sqrt{x^2-1})$. Applying this to (34.12), plus a lot of computation, shows the equivalence to our original distance formula.

Maybe a better way to understand distance is to look at circles. The *hyperbolic circle with center z and radius r* is naturally defined as the set of all points w such that $\mathrm{dist}(z,w) = r$. Let's wrestle with this equation. It means that

$$\begin{aligned}
&\frac{|z-w|^2}{2\,\mathrm{im}(z)\,\mathrm{im}(w)} = \cosh(r) - 1 \\
\Leftrightarrow\ &(\mathrm{re}(z)-\mathrm{re}(w))^2 + (\mathrm{im}(z)-\mathrm{im}(w))^2 = 2(\cosh(r)-1)\mathrm{im}(z)\mathrm{im}(w) \\
\Leftrightarrow\ &(\mathrm{re}(z)-\mathrm{re}(w))^2 + (\cosh(r)\mathrm{im}(z)-\mathrm{im}(w))^2 = \cosh(r)^2 - 1 = \sinh(r)^2\mathrm{im}(z)^2.
\end{aligned} \tag{34.13}$$

Writing $x = \mathrm{re}(z)$, $y = \mathrm{im}(z)$, the outcome is:

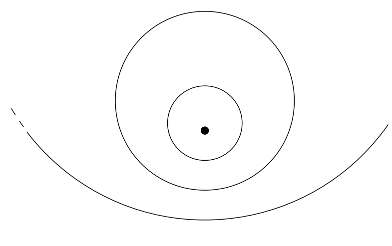
FACT 34.5. *The hyperbolic circle with center* (x, y) *and radius* r *is exactly the ordinary Euclidean circle with center* $(x, \cosh(r)y)$ *and radius* $\sinh(r)y$.

An easy check, as well as a way to remember this, is: the top and bottom points of the circle are

$$(x, \cosh(r)y + \sinh(r)y) = (x, e^r y) \quad \text{and} \quad (x, \cosh(r)y - \sinh(r)y) = (x, e^{-r}y). \tag{34.14}$$

Indeed, as we see from Example 34.4, both points have hyperbolic distance r from (x, y). Here is a picture of hyperbolic circles centered at $i = (0, 1)$:

(34.15)



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