35. Arclengths and areas

So far, concepts of hyperbolic geometry, such as geodesics and distances, have appeared in somewhat haphazard and unrelated ways. The modern viewpoint is that these, and another notions, arise from a single concept, that of the infinitesimal length element.

- The infinitesimal length element appears most naturally in the notion of hyperbolic arclength of a path. We introduce that, and explain how it gives rise to hyperbolic geodesics and to the distance formula.
- We also consider the associated notion of hyperbolic area, and explain how that can be used to improve our understanding of hyperbolic triangles.

(35a) Lengths and distances. At a point (x, y) in the hyperbolic plane, the *infinitesimal length element* is

$$\frac{\sqrt{dx^2 + dy^2}}{y}. \tag{35.1}$$

In terms of complex numbers $z = x + iy$, one would write that formula as

$$\frac{|dz|}{\mathrm{im}(z)}, \tag{35.2}$$

where $\mathrm{im}(z) = y$ is the imaginary part. The infinitesimal length element describes how the geometry is distorted, compared to the Euclidean one, around this particular point. Our first use of this is to define the arclength of a path $c(t) = (x(t), y(t)) \in \mathbb{H}$:

$$\mathrm{length}(c) = \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt. \tag{35.3}$$

If we think of our path as complex-valued, $c(t) \in \mathbb{C}$, the same formula is

$$\mathrm{length}(c) = \int \frac{|c'(t)|}{\mathrm{im}(c(t))}\, dt. \tag{35.4}$$

Here is how arclength leads one naturally to the concepts of distance and of geodesic:

Theorem 35.1. *For any two points z and w,*

$$\mathrm{dist}(z, w) = \min\{\mathrm{length}(c) \textit{ for all paths } c \textit{ from } z \textit{ to } w\}. \tag{35.5}$$

Theorem 35.2. *Given two points z and w, the paths of minimal length from z to w are precisely those that go along a geodesic (without ever turning back, of course).*

Let's prove these theorems in a special case, namely $z = (0, 1)$ and $w = (0, e^r)$, where $r > 0$. We know from the distance formula that $\mathrm{dist}(z, w) = r$. Take any path c from z to w. Then

$$\begin{aligned} \mathrm{length}(c) &= \int \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{y(t)}\, dt \\ &\geq \int \frac{\sqrt{(dy/dt)^2}}{y(t)}\, dt \geq \int \frac{dy/dt}{y(t)} dt = \ln(y(t))\Big|_{\text{starting } t}^{\text{endpoint } t} = \ln(e^r) - \ln(1) = r. \end{aligned} \tag{35.6}$$

So, the length is indeed always $\geq$ the distance. Moreover, in order for (35.6) to be an equality, we must have $dx/dy = 0$ and $dy/dt \geq 0$ everywhere. So, minimal length paths are those that move upwards along the vertical line.

We haven't been too precise about the class of paths that is allowed. Basically, anything for which you can carry out the arclength integral works, let's say a piecewise smooth path.

COROLLARY 35.3. *For any three points z, u, w, we have*

$$\mathrm{dist}(z, w) \leq \mathrm{dist}(z, u) + \mathrm{dist}(u, w). \tag{35.7}$$

We could of course prove this directly from the definition of distance (ouch), but it follows much more easily from Theorem 35.1: take a length-minimal path from z to u, and another such path from u to w. Together, they give a path from z to w of length $\mathrm{dist}(z, u) + \mathrm{dist}(u, w)$. That sum must therefore be $\geq \mathrm{dist}(z, w)$.

(35b) Area. If infinitesimal lengths are stretched by $1/y$, areas should be stretched by $1/y^2$. Indeed, we define the hyperbolic area of a region $U \subset \mathbb{H}$ by

$$\mathrm{area}(U) = \int_U \frac{1}{y^2}\, dx\, dy. \tag{35.8}$$

EXAMPLE 35.4. *Suppose that our region is the area between two graphs $y = q(x)$ and $y = p(x)$:*

$$U = \big\{a \leq x \leq b,\ q(x) \leq y \leq p(x)\big\} \tag{35.9}$$

Then, we can carry out the area computation by integrating y first, just like you learned in calculus:

$$\mathrm{area}(U) = \int_a^b \Big(\int_{f(x)}^{g(x)} \frac{1}{y^2}\, dy\Big)\, dx = \int_a^b \frac{1}{p(x)} - \frac{1}{q(x)}\, dx. \tag{35.10}$$

EXAMPLE 35.5. *Take the region bounded by the geodesics $x = -1$, $x = +1$ and $x^2 + y^2 = 1$ (this looks like a triangle, but it's not, since the sides don't actually meet in $\mathbb{H}$). We get an improper integral*

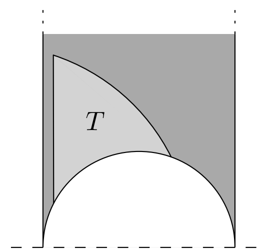
$$\int_{-1}^1 \Big(\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\Big)\, dx = \int_1^r \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x)\Big|_{x=-1}^{x=1} = \pi. \tag{35.11}$$

One can consider any region of the same shape, bounded by a half-circle and two vertical half-lines which are asymptotic to the same points on the horizontal axis. The same argument (with a small change of variables) shows that the area is always π.

THEOREM 35.6. *The area of any hyperbolic triangle is $< \pi$.*

Let's look at a special case, which is when one side of the triangle is a vertical line. This is actually straightforward, since the triangle is clearly contained in a larger region of area π:

(35.12)



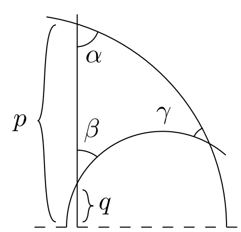
(35c) Triangle geometry. We know two things about hyperbolic triangles: first, the sum of the angles is $< \pi$; and second, the area is $< \pi$. The two are actually related:

THEOREM 35.7. *(Gauss-Bonnet) For a hyperbolic triangle T, with angles (α, β, γ),*

$$\text{area}(T) = \pi - \alpha - \beta - \gamma. \tag{35.13}$$

As usual, we will just look at the case where one of the sides is vertical,

(35.14)



What happens if we move that side? The fundamental theorem of calculus, applied to the area integral, tells us that

$$\tfrac{d}{dx}\text{area}(T) = \tfrac{1}{p} - \tfrac{1}{q}, \tag{35.15}$$

where x is the coordinate giving the position of the vertical line; and last time we saw that

$$\tfrac{d}{dx}(\alpha + \beta + \gamma) = \tfrac{1}{q} - \tfrac{1}{p}. \tag{35.16}$$

As a consequence,

$$\tfrac{d}{dx}(\text{area}(T) - \pi + \alpha + \beta + \gamma) = 0, \tag{35.17}$$

which means that expression in brackets is constant in x. If we move the vertical line to the right until the triangle shrinks to a point, then in the limit $\alpha + \beta + \gamma \to \pi$ and $\text{area}(T) \to 0$, so $(\text{area}(T) - \pi + \alpha + \beta + \gamma) \to 0$. A constant function which has limit 0 is of course 0 everywhere, and that's what we wanted!

We close this chapter by stating two formulae without proof. Take a hyperbolic triangle with side-lengths (a, b, c) and angles (α, β, γ), labeled in the way you're used to from ordinary geometry.

The hyperbolic cosine laws are

$$\cos(\alpha) = \frac{\cosh(b)\cosh(c) - \cosh(a)}{\sinh(b)\sinh(c)}, \tag{35.18}$$

$$\cosh(a) = \frac{\cos(\alpha) + \cos(\beta)\cos(\gamma)}{\sin(\beta)\sin(\gamma)}. \tag{35.19}$$

The first cosine law recovers the angles in a triangle from the three side-lengths, which is something one can also do in Euclidean geometry. The second law, on the other hand, would be inconceivable in Euclidean geometry, where one certainly can't determine side-lengths starting only with angles, because of the freedom to scale up any triangle! As one already sees from our discussion of the maximal area of triangles, there's no "scaling up" in hyperbolic geometry.

At this point, we need to own up to the sins we've committed:

- we have established the key relation between arclength and distance (Theorems 35.1 and 35.2) only when $z = (0, 1)$ and $w = (0, e^r)$, $r > 0$.
- we have proved Gauss-Bonnet (both its proper form, Theorem 35.7, and the preliminary Theorem 35.6) only for triangles where one side is a vertical line.

In both situations, the general statement actually follows from those special cases, but that will have to wait until the next lecture.