36. Hyperbolic isometries

One way to think of Euclidean geometry is that the basic notion is congruence transformations: all geometric notions must be invariant under them. There is a similar class of transformations in hyperbolic geometry.

- We introduce those transformations, called hyperbolic isometries, both through special classes, and a general formalism.
- Often, isometries can be applied to simplify a coordinate argument or computation. We want to at least lay the groundwork for that.

(36a) The simplest isometries. A hyperbolic isometry is a map $\Phi : \mathbb{H} \to \mathbb{H}$ which is reversible, and compatible with all notions of hyperbolic geometry we have introduced. Namely, it:

(1) preserves angles (in the sense of the angle between the tangent lines, at the intersection of two curves);

(2) preserves hyperbolic distances;

(3) preserves hyperbolic arclengths of curves;

(4) preserves hyperbolic areas of regions;

(5) takes hyperbolic geodesics to hyperbolic geodesics;

(6) takes hyperbolic circles to hyperbolic circles.

These are not all logically independent. For instance, there's an implication (2) $\Rightarrow$ (6) (because circles are defined in terms of distances). All the transformations which we'll discuss have those properties, even though we won't verify them.

Let's introduce some special examples of hyperbolic isometries. The simplest one are (horizontal) *translations*:

$$\Phi(x, y) = (x + b, y), \quad \text{where } b \text{ is some real number.} \tag{36.1}$$

Vertical translations can't be allowed. For starters, a vertical translation is not an invertible map from the upper half-plane to itself; and even if we somehow ignored that, it fails to have the required properties. Instead, we have (radial) *rescalings*:

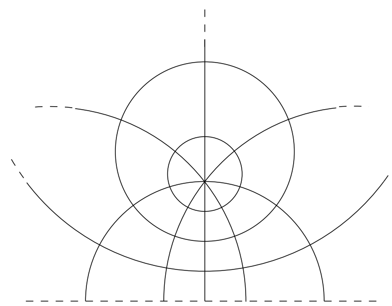
$$\Phi(x, y) = (ax, ay), \quad \text{where } a \text{ is some positive real number.} \tag{36.2}$$

(36b) Hyperbolic rotations. Clearly, there should be some analogue of rotations. Ordinary Euclidean rotations won't do, because they don't take $\mathbb{H}$ to $\mathbb{H}$. Our first intuition is that a rotation should have a center that's fixed. Let's just look at rotations with center $i = (0, 1)$. These should be hyperbolic isometries that satisfy $\Phi(i) = i$. What other properties should such a transformation have?

- Since it fixes i and preserves distances, each hyperbolic circle with center i must be preserved under Φ.
- It must send any geodesic passing through i to another such geodesic.
- Presumably, to deserve the name, Φ should rotate tangent directions at the point i by some (anticlockwise) angle, let's call it θ.

What we are saying is that a hyperbolic rotation should preserve the "hyperbolic polar coordinate system" given by geodesics through i and hyperbolic circles centered at i:

(36.3)



This idea, together with the fact that the derivative of Φ at the point i is a rotation by θ, completely describes the transformation. There's also a formula: the hyperbolic rotation with center i and angle θ is, in complex coordinates,

$$\Phi(z) = \frac{\cos(\theta/2)z + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)z}. \tag{36.4}$$

While this formula is not easy to parse, it certainly satisfies

$$\Phi(i) = \frac{\cos(\theta/2)i + \sin(\theta/2)}{\cos(\theta/2) - i\sin(\theta/2)} = \frac{i(\cos(\theta/2) - i\sin(\theta/2))}{\cos(\theta/2) - i\sin(\theta/2)} = i. \tag{36.5}$$

A computation of the derivative (Jacobian) of Φ at the point $z = i$ shows that it really rotates tangent lines by θ; but we'll have to omit that.

EXAMPLE 36.1. *The simplest example of a hyperbolic rotation occurs for $\theta = \pi$. This is called an inversion:*

$$\Phi(z) = -\frac{1}{z} = -\frac{\bar{z}}{|z|^2}, \tag{36.6}$$

or in real coordinates,

$$\Phi(x, y) = \frac{(-x, y)}{x^2 + y^2}. \tag{36.7}$$

It takes $\{x^2 + y^2 = r\}$ to $\{x^2 + y^2 = 1/r\}$ for any r, hence the name; and it also flips the sign of the x-coordinate.

(36c) What we can do with them. Isometries can often be used to bring geometric objects into a position where they are more convenient for coordinate computations. This may involve one of the transformations listed above, or more typically the composition of several of them.

FACT 36.2. *Given two points z and w, there is an isometry Φ such that $\Phi(z) = w$.*

Let's first assume $w = i = (0,1)$. One can rescale z so that its y-coordinate becomes 1, and then use horizontal translation to move it to $(0,1)$, so that works. How about the general case? Well, first we move z to $(0,1)$, and then we use the reverse of that argument to move $(0,1)$ to w.

FACT 36.3. *Given two geodesics c and d, there is an isometry Φ such that $\Phi(c) = d$.*

Let's pick some point w on our geodesic c. We can find a first isometry that moves that point to $(0,1)$. After applying that, our geodesic will be one that goes through $(0,1)$. By applying a hyperbolic rotation, we can rotate the tangent line at $(0,1)$ of our geodesic so it points vertically. Since a geodesic is determined by one point and the tangent line at that point, the entire geodesic ends up being the y-axis. This argument proves that we can transform c to the y-axis, and using the reverse of that, we can then map the y-axis to d.

FACT 36.4. *Given any hyperbolic triangle, there is an isometry Φ such that after applying that isotopy, one of the vertices of the triangle is i, and the other vertex is $e^r i$ for some $r > 0$.*

That almost follows from what we've said. We can pick a side of the triangle, and use an isometry to take that to the y-axis. Then, a suitable rescaling takes the bottom vertex to $(0,1)$ and the top vertex to $(0,e^r)$, as desired.

As an example of how these arguments can be useful, remember our theorem from last time, that the hyperbolic distance between two points is the minimal arclength of a path connecting them, and that length-minimizing paths follow geodesics. We only proved this for points of the form $(0,1)$ and $(0,e^r)$. But by applying an isometry (which preserves both distance and arclength), the general case follows. The same idea applies to Gauss-Bonnet, which was the other issue listed at the end of the previous lecture.

(36d) The general formula. Here's the general form of hyperbolic isometries. Take a real 2×2 matrix A, with $\det(A) > 0$. Each such matrix gives rise to a hyperbolic isometry A, like this:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{we have} \quad \Phi_A(z) = \frac{az+b}{cz+d}. \tag{36.8}$$

Note that A and λA, for any $\lambda \neq 0$, give the same transformation. So there are really three (not four) degrees of freedom in choosing Φ_A. The previously discussed classes of transformations are all special cases. For (horizontal) translations, take

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \implies \Phi_A(z) = z + b \tag{36.9}$$

For (radial) expansions,

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \implies \Phi_A(z) = (a/d)z \tag{36.10}$$

with $ad > 0$ (as far as Φ_A is concerned, only the quotient $a/d > 0$ matters). The case of hyperbolic rotations (with center i) is:

$$A = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \implies \Phi_A(z) = \frac{\cos(\theta/2)z + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)z}. \tag{36.11}$$

It may be a little weird that the hyperbolic rotation that rotates tangent lines at i by θ is given by the Euclidean rotation matrix A with angle $-\theta/2$; but that's just a clash of conventions, nothing to get excited about.

The reason for writing the parameters as matrix entries is that composition of isometries is governed by matrix multiplication,

$$\Phi_A(\Phi_B(z)) = \Phi_{AB}(z). \tag{36.12}$$

The identity matrix gives the identity (trivial) isometry, and therefore, the inverse matrix gives the inverse isometry; so the class of Φ_A is closed under composition as well as passing to inverses. This ease in writing down compositions is precisely the advantage of the matrix framework.

EXAMPLE 36.5. *Suppose that we want to have hyperbolic rotations centered not at i, but at some arbitrary point $z = (x, y)$. We can achieve that by composing three elements:*

- *First, we use an initial isometry to move z to i. One can construct this out of translations and expansions (Fact 36.2). As a formula:*

$$B = \begin{pmatrix} 1 & -x/y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}. \tag{36.13}$$

- *Then, we take the matrix A from* (36.11) *to do the rotation centered at i;*
- *Finally, we use B^{-1} to move i back to z.*

The outcome is that our desired rotation-with-center-z is given by the matrix (we multiply by y to simplify the formula, that doesn't affect the isometry)

$$yB^{-1}AB = \begin{pmatrix} y\cos(\theta/2) - x\sin(\theta/2) & (x^2+y^2)\sin(\theta/2) \\ -\sin(\theta/2) & y\cos(\theta/2) + x\sin(\theta/2) \end{pmatrix}. \tag{36.14}$$

Strictly speaking, we have only covered half of the symmetries of the hyperbolic plane: reflection along the y-axis, $R(x, y) = (-x, y)$, should also be allowed. A general symmetry is then either Φ_A or $R\Phi_A$. However, for most purposes the Φ_A are enough, and their formalism is particularly satisfying.