37. The geodesic equation

In hyperbolic geometry, lengths and distances are changed from their Euclidean counterparts. This then affects the notion of straight line (geodesic). At the root of those changes lies the length element $y^{-1}\sqrt{dx^2+dy^2}$. What happens if instead of y^{-1}, one takes a general (positive) function of (x,y)? This gives a lot of different "curved geometries". What are geodesics in each of those? We will get to the answer by:

- introducing a differential equation for parametrized curves (the geodesic equation);
- examining the geometric meaning of that equation; and
- for hyperbolic geometry, checking that the solutions agree with our previous definition of geodesic.

(37a) Curved geometries. A curved geometry is specified by its infinitesimal length element

$$e^{\psi(x,y)}\sqrt{dx^2+dy^2}, \tag{37.1}$$

where $\psi(x,y) \in \mathbb{R}$ can be any function. Hyperbolic geometry is one example, with $\psi(x,y) = -\ln(y)$. The factor $e^{\psi(x,y)}$ tells us how much the geometry is "bunched up" around (x,y). As before, the notion of arclength gets modified accordingly:

$$\operatorname{length}(c) = \int e^{\psi(c(t))}\|c'(t)\|\,dt. \tag{37.2}$$

If you think of $c(t)$ as a point moving in time, then $e^{\psi(c(t))}\|c'(t)\|$ is its speed with respect to our curved geometry.

Definition 37.1. *The geodesic equation for a curve $c(t) \in \mathbb{R}^2$ is*

$$c''(t) - \|c'(t)\|^2\,\nabla\psi_{c(t)} + 2(\nabla\psi_{c(t)}\cdot c'(t))\,c'(t) = 0. \tag{37.3}$$

There are always the boring stationary solutions $c(t) = constant$. The other solutions define curves that we call the geodesics for our geometry.

In the Euclidean plane ($\psi(x,y) = 0$) the equation reduces to $c''(t) = 0$, which is Newtonian motion with no force applied; the solutions are $c(t) = vt + w$ for $v, w \in \mathbb{R}^2$, meaning straight-line motion at constant speed. For a general curved geometry, the geodesic equation yields precisely the analogue of straight-line motion.

(37b) The geodesic equation in (x,y)-components. The geodesic equation may not look particularly appealing, but we'll get used to it and its properties. It is an equality of vectors, which we can separate into components $c(t) = (x(t), y(t))$. One has $\|c'(t)\|^2 = x'(t)^2 + y'(t)^2$, and $\nabla\psi = (\partial_x\psi, \partial_y\psi)$. After some cancellations, the component equations are:

$$\begin{aligned} &x''(t) + (x'(t)^2 - y'(t)^2)\partial_x\psi + 2\,x'(t)y'(t)\partial_y\psi = 0,\\ &y''(t) + (y'(t)^2 - x'(t)^2)\partial_y\psi + 2\,x'(t)y'(t)\partial_x\psi = 0. \end{aligned} \tag{37.4}$$

Here, the partial derivatives of ψ are taken at the point $(x(t), y(t))$.

EXAMPLE 37.2. *Let's look at hyperbolic geometry. There, the equations become*

$$\begin{aligned} & x''(t) - 2\frac{x'(t)y'(t)}{y(t)} = 0, \\ & y''(t) + \frac{x'(t)^2 - y'(t)^2}{y(t)} = 0. \end{aligned} \tag{37.5}$$

A useful trick is to rewrite them as

$$\begin{aligned} & \frac{d}{dt}\frac{x'(t)}{y(t)^2} = 0, \\ & \frac{d}{dt}\frac{x'(t)x(t) + y'(t)y(t)}{y(t)^2} = 0. \end{aligned} \tag{37.6}$$

We can integrate,

$$\begin{aligned} & x'(t) = Ay(t)^2, \\ & x'(t)x(t) + y'(t)y(t) = By(t)^2, \end{aligned} \tag{37.7}$$

for some constants A, B. For $A = 0$, we have $x'(t) = 0$, so $(x(t), y(t))$ moves along a vertical line. What if $A \neq 0$? We combine the two equations,

$$x'(t)x(t) + y'(t)y(t) = (B/A)x'(t) \quad \Leftrightarrow \quad (x(t) - B/A)x'(t) + y(t)y'(t) = 0, \tag{37.8}$$

and then integrate again:

$$(x(t) - B/A)^2 + y(t)^2 = C. \tag{37.9}$$

This describes the circle of radius $\sqrt{C}$ centered at the point $(B/A, 0)$ on the horizontal axis. In this way, we have recovered all the hyperbolic geodesics.

(37c) The geodesic equation in components parallel and orthogonal to the motion. Of course, (x, y)-coordinates are somewhat arbitrary. A better approach is to separate the geodesic equation into a component that is a multiple of $c'(t)$, and a component which is orthogonal to it. This is done by taking the dot product and the cross product of the equation with $c'(t)$ (for our notion of cross product of vectors in $\mathbb{R}^2$, which produces a number). The outcome, again after some cancellation, is:

$$(c''(t) + \|c'(t)\|^2\, \nabla\psi_{c(t)}) \cdot c'(t) = 0, \tag{37.10}$$

$$(c''(t) - \|c'(t)\|^2\, \nabla\psi_{c(t)}) \times c'(t) = 0. \tag{37.11}$$

FACT 37.3. *Along a solution $c(t)$ of the geodesic equation, the speed (with respect to our curved geometry) $e^{\psi(c(t))}\|c'(t)\|$ is constant.*

This is equivalent to saying that $e^{2\psi(c(t))}\|c'(t)\|^2$ is constant. If we differentiate that, we get

$$\frac{d}{dt}\big(e^{2\psi(c(t))}\|c'(t)\|^2\big) = 2e^{2\psi(c(t))}\big(c''(t) + \|c'(t)\|^2\, \nabla\psi_{c(t)}) \cdot c'(t)\big), \tag{37.12}$$

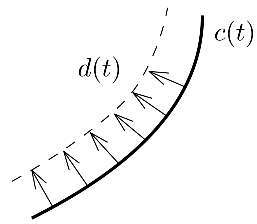
which is zero by (37.10). Physically, think of this (after dividing by 2) as conservation of kinetic energy, $\frac{1}{2}\text{speed}^2$, which is a feature of Newtonian motion.

The other equation (37.11) is more crucial geometrically, since it determines how a geodesic bends, but harder to explain. If we have a curve $c(t)$ with $c'(t) \neq 0$, we can look at ones that have been displaced sideways with respect to the tangent direction, by which we mean

$$d(t) = c(t) + \delta\, Jc'(t). \tag{37.13}$$

where we have used the 90° rotation matrix $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and δ is a small constant, which governs the amount and direction (left or right) of displacement.

(37.14)



LEMMA 37.4. *Suppose that $c'(t)$ is never zero. Then the arclength integrand for $d(t)$ is approximately (to first order in δ, which means ignoring terms that are quadratic or higher order in δ and δ') given by*

$$e^{\psi(d)}\|d'(t)\| \approx e^{\psi(c)}\Big(\|c'(t)\| + \frac{\delta}{\|c'(t)\|}(c''(t) - \|c'(t)\|^2\nabla\psi) \times c'(t)\Big). \tag{37.15}$$

I will spare you the gory details. The important point is that what appears in (37.15) is $\delta/\|c'(t)\|$ times the second part of the geodesic equation. In Euclidean geometry, whenever $c(t)$ curves to the left, displacing it in that direction makes it shorter, and similar for curving to the right. Straight lines are characterized by the fact that the displaced version proceed at the same speed as the original curve. In our context, the same characterization of geodesics is at least approximately true (to first order in δ).

THEOREM 37.5. *Suppose that $c(t)$, $t \in [a,b]$, is a curve that proceeds with constant speed in our geometry. Suppose also that among all curves connecting the endpoints $c(a)$ and $c(b)$, ours achieves the minimal possible arclength. Then $c(t)$ is a solution of the geodesic equation.*

As before, we consider displaced curves, but where the amount of displacement $\delta(t)$ is now a function, satisfying $\delta(a) = \delta(b) = 0$. This condition ensures that the displaced curve $d(t)$ has the same endpoints as $c(t)$. By assumption, $\text{length}(c) \geq \text{length}(d)$, which for small displacements (at first order) implies that

$$\int_a^b \delta(t)\big((c'' - \|c'\|^2\nabla\psi) \times c'\big)\,dt \geq 0. \tag{37.16}$$

This must hold for all possible $\delta(t)$. In particular, we can take

$$\delta(t) = -f(t)\big((c'' - \|c'\|^2\nabla\psi) \times c'\big), \tag{37.17}$$

where $f(t)$ is a function defined for $t \in [a,b]$, such that $f(a) = f(b) = 0$, and all other values $f(t)$ are positive. In that case, what (37.16) says is that

$$\int_a^b -f(t)\big((c'' - \|c'\|^2\nabla\psi) \times c'\big)^2\,dt \geq 0. \tag{37.18}$$

There's an almost-contradiction here: the integrand is ≤ 0, and the integral is supposed to be ≥ 0. The only way out is that the integrand is actually zero. But since $f(t) > 0$ for $t \in (a, b)$, this means that $(c'' - \|c'\|^2 \nabla\psi) \times c'$ must be zero, which is (37.11) (the other equation (37.10) is already true, because of the assumption that c advances with constant speed).