38. Behaviour(s) of geodesics

In the last lecture, we encountered the geodesic equation for general curved geometries, but the only concrete example we considered was hyperbolic geometry, where we already knew what the outcome would be. This time, we'll proceed along two lines of enquiry:

- we get more experience with the general properties of solutions of the geodesic equation;
- and we look at special classes of geometries where solutions become easier to find.

The guiding question is to what extent solutions, in a general geometry, share (or do not share) the qualitative behaviour of straight lines in Euclidean geometry.

(38a) The geodesic equation, revisited. We decided that in a curved geometry

$$e^{\psi(x,y)}\sqrt{dx^2+dy^2}, \tag{38.1}$$

the analogue of straight-line motion should be a curve $c(t) \in \mathbb{R}^2$ which solves

$$c'' - \|c'\|^2 \nabla\psi + 2(\nabla\psi \cdot c')c' = 0. \tag{38.2}$$

We can look at this from a pure differential equations viewpoint, and check some basic properties:

- If $c(t)$ is a geodesic, then so is $c(t+T)$ for any constant T. In other words, there's no geometrically preferred "starting point" on a geodesic.
- If $c(t)$ is a geodesic, then so is $c(Rt)$ for any constant R (including a negative one). To see that, note that the first derivatives of $c(Rt)$ scale linearly with R, and the second derivatives quadratically. But since the equation has quadratic terms in $c'(t)$, everything works out.
- Given a starting point and a starting velocity vector, there is one and only one geodesic c such that $c(0)$ is our starting point and $c'(0)$ is our starting velocity vector. (This is a general property of second order differential equations, and the only time that we'll use the theory of such equations.)

Combining all three properties, one sees the following:

FACT 38.1. *If two geodesics become tangent at some point, then they trace out the same curve (in fact, up to reparametrizations $c(Rt+T)$, the two geodesics are the same).*

(38b) Translationally invariant geometries. Suppose that our geometry depends only on one of the variables, $\psi = \psi(y)$. The geodesic equations simplify to

$$\begin{aligned} &x'' + 2\,x'y'(d\psi/dy) = 0, \\ &y'' + ((y')^2 - (x')^2)\,(d\psi/dy) = 0, \end{aligned} \tag{38.3}$$

There are a few special solutions which are easy to describe.

FACT 38.2. *Any vertical line can be parametrized so that it becomes a geodesic:*

$$x(t) = C,\ \ y(t) \text{ a solution of } y'' + (y')^2(d\psi/dy) = 0. \tag{38.4}$$

FACT 38.3. *If the derivative $d\psi/dy$ is zero (ψ has a critical point) for some value of y, then that particular horizontal line (with $x(t) = A + Bt$) is also a geodesic.*

We can gain some understanding of more general solutions, but that requires digging deeper into the equations. The first line of (38.3) can be written as

$$\frac{d}{dt}\big(e^{2\psi}x'\big) = 0. \tag{38.5}$$

This generalizes something we've seen for hyperbolic geometry, namely the first equation in (37.6). There is an underlying principle from classical mechanics, where any symmetry gives rise to a quantity that's constant under the motion. In our case, the symmetry of the geometry under horizontal translations gives rise to the invariant quantity $e^{2\psi}x'$, which is the horizontal component of momentum. Speaking of conserved quantities, we also have conservation of speed (which, in mechanics, arises from the time-translation invariance of the geodesic equation). The two conserved quantities give us the equations

$$\begin{aligned} x' &= e^{-2\psi}A, \\ (x')^2 + (y')^2 &= e^{-2\psi}B, \end{aligned} \tag{38.6}$$

with constants A and B. Inserting them into the second line of (38.3) gives us

$$y'' + (e^{-2\psi}B - 2e^{-4\psi}A^2)(d\psi/dy) = 0. \tag{38.7}$$

This kind of equation, which we can write as $y'' = -dU/dy$ with

$$U(y) = \int (e^{-2\psi}B - 2e^{-4\psi}A^2)(d\psi/dy)\, dy = -\frac{B}{2}e^{-2\psi(y)} + \frac{A^2}{2}e^{-4\psi(y)}, \tag{38.8}$$

has a well-known meaning. It describes the motion of a Newtonian particle $y(t)$ (with one degree of freedom) in the potential $U(y)$, like someone skiing over a hill of shape $U(y)$. Note however that $U(y)$ depends on A and B – and those depend on which specific geodesics we are considering: they can be read off from the starting position and velocity at any specific time.

When doing this analysis, we had one particular kind of behaviour in mind. Suppose that we are at a local maximum of the function which defines the curved geometry. For simplicity, let's say that the maximum happens at $y = 0$, and has the form

$$\psi(0) = 0,\ \psi'(0) = 0,\ \psi''(0) < 0. \tag{38.9}$$

We know then that the x-axis with its standard parametrization ($x(t) = t$, $y(t) = 0$) is a geodesic. What happens if we start at a point on the x-axis, but with a starting direction that's not quite horizontal? So

$$\begin{aligned} &x'(0) = 1, \\ &y(0) = 0; \\ &y'(0)^2 = \epsilon \ \text{ for some } \epsilon. \end{aligned} \tag{38.10}$$

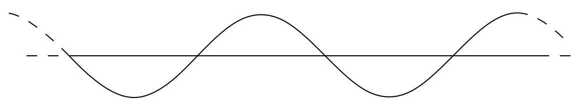
By plugging that into (38.6), we see that $A = 1$, $B = 1 + \epsilon$. The potential is therefore

$$U(y) = -\frac{1+\epsilon}{2} e^{-2\psi(y)} + \frac{1}{2} e^{-4\psi(y)}. \tag{38.11}$$

It is easy to see that $U'(0) = 0$. A more laborious computation shows that

$$U''(0) = -(1+\epsilon)\psi''(0) < 0 \tag{38.12}$$

So the potential has a minimum at $y = 0$. If ϵ is small, our picture is therefore that of a particle oscillating in the bottom of a trough. This means that $y(t)$ will perform small oscillations around 0, while $x'(t) = e^{-2\psi(y)}$ means that $x(t)$ keeps moving to the right at a speed approximately equal to 1. Our geodesic will behave like this:



(38.13)

(It's not a simple sin-wave, even though qualitatively it looks the same.) From that, we take away the following general insight:

FACT 38.4. *In general, it is possible for two geodesics to intersect each other more than once; in fact, they can intersect infinitely many times.*

(38c) Rotationally invariant geometries. Suppose that our geometry is invariant under rotations around the origin, which means that $\psi = \psi(r)$ can be written as a function of $r = \sqrt{x^2+y^2}$. We could analyze this as before, but there's a change of coordinate trick that we can do instead, and which saves us a lot of time. Namely, use the following version of polar coordinates:

$$\begin{aligned} x &= e^{\rho}\cos(\theta), \\ y &= e^{\rho}\sin(\theta). \end{aligned} \tag{38.14}$$

A computation, which we omit, shows the following:

PROPOSITION 38.5. $(x(t), y(t))$ *is a geodesic for our rotationally invariant geometry if and only if* $(\theta(t), \rho(t))$ *is a geodesic for the geometry* $\psi(e^{\rho}) + \rho$.

If we think of (θ, ρ) as Cartesian coordinates in a plane, then $\psi(e^{\rho}) + \rho$ is a translation-invariant geometry. For such geometries, we know (Fact 38.2) that all lines $\theta = \textit{constant}$ are geodesics, and that implies the following:

FACT 38.6. *In a rotationally invariant geometry, all radial lines (straight lines through the origin), parametrized in an appropriate way, are geodesics.*

In (θ, ρ) coordinates, we also know (Fact 38.3) that a line $\rho = \textit{constant}$ is a geodesic if, at that value of ρ, the function defining the geometry has a critical point. In our case, that condition says that

$$\tfrac{d}{d\rho}(\psi(e^{\rho}) + \rho) = 0 \;\;\Leftrightarrow\;\; \psi'(e^{\rho}) = -1/e^{\rho}. \tag{38.15}$$

Bearing in mind that $e^{\rho} = r$ is the radius, we find that:

FACT 38.7. *In a rotationally invariant geometry, the circle of radius $r > 0$ around the origin is a geodesic if and only if $\psi'(r) = -1/r$ (at that value of r).*

It is easy to construct geometries for which this equation has solutions (indeed, you may have already seen one in Problem 37.2, even though there, we did not use radial coordinates). This illustrates:

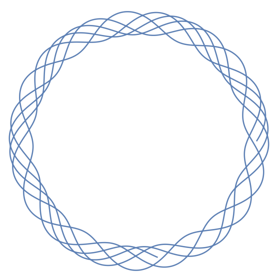
FACT 38.8. *In general, it is possible for a geodesic to come back to its starting point, and even to be periodic.*

By dividing such a circle into a small and a large piece, we see:

FACT 38.9. *In general, it is possible for a segment of a geodesic to not be the shortest path between its endpoints.*

In our previous discussion of translationally-invariant geometries we found that under certain assumptions, if a horizontal line is a geodesic, there are other geodesics that oscillate around it. In the rotationally-invariant case, that translates into geodesics that oscillate around circles:

(38.16)



In general, the oscillation is by no means guaranteed to return to the initial position after going once, or even several times, round the circle. Instead, the geodesic could, as it goes, gradually weave a denser and denser web of oscillations around the circle, without ever repeating. We learn:

FACT 38.10. *It is possible for a geodesic to cross itself, even infinitely many times.*