39. Curvature

In this lecture, we look at the quantity that gives a precise meaning to the expression "curved geometries", namely the (Gauss) curvature.

- We define curvature, and compute a few simple examples.
- It's a good idea to average (integrate) the curvature over regions of our geometry. This leads to a general version of the Gauss-Bonnet theorem.

(39a) From geodesics to curvature. We saw that the geodesic equation can be thought of as consisting of two parts:

(39.1) along a geodesic $c(t)$, the speed $e^{\psi(c(t))}\|c'(t)\|$ is constant;

(39.2) the geodesic tries to "not turn left or right", $(c''(t) - \|c'(t)\|^2 \nabla\psi_{c(t)}) \times c'(t) = 0$.

While these equations are hard to solve, we can suppose that we know one (nonconstant) solution $c(t)$, and study the behaviour of other solutions which are close by. Specifically, let's take sideways displacement by a varying amount $\delta(t)$,

$$d(t) = c(t) + \delta(t)Jc'(t), \tag{39.3}$$

where J is the matrix that rotates vectors by 90°. Look at (39.2) for $d(t)$, and think of $\delta(t)$ as small, so that all terms which are quadratic or higher order in $\delta(t)$ can be omitted. The outcome, after a lot of computation which we skip, is the *approximate geodesic equation*

$$\delta''(t) - \|c'(t)\|^2 (\Delta\psi)_{c(t)}\, \delta(t) = 0, \tag{39.4}$$

where $\Delta\psi = \partial_x^2\psi + \partial_y^2\psi$ is the Laplace operator (which we've also seen in Chapter III). It contains an important geometric quantity:

DEFINITION 39.1. *The curvature of a curved geometry is the function*

$$K = -e^{-2\psi}\Delta\psi. \tag{39.5}$$

If the geodesic proceeds with unit speed in our geometry, meaning that $e^{\psi(c(t))}\|c'(t)\| = 1$, we can write (39.4)

$$\delta''(t) + K(c(t))\delta(t) = 0. \tag{39.6}$$

This equation describes the approximate behaviour of geodesics that are close to our original $c(t)$. For instance, suppose that we have constant positive curvature $K = C > 0$. Then (39.6) becomes $\delta'' + C\delta = 0$, which has the solutions $\delta(t) = A\sin(\sqrt{C}t) + B\cos(\sqrt{C}t)$. Hence, in this case we see nearby geodesics that oscillate around the original one (as a qualitative conclusion, this is correct, in spite of the fact that we're looking at an approximation to the geodesic equation).

EXAMPLE 39.2. *Hyperbolic geometry, $\psi = -\ln(y)$, has constant negative curvature:*

$$K = -y^2\Delta(-\ln(y)) = -1. \tag{39.7}$$

EXAMPLE 39.3. *Take*

$$\psi(y) = -\ln(\cosh(y)). \tag{39.8}$$

This is the length element for a round sphere parametrized by the Mercator map-making projection. The curvature is constant, but with the opposite sign.

$$K = \cosh(y)^2(d^2/dy^2)(\ln(\cosh(y))) = \cosh(y)^2(d/dy)\tanh(y) = 1. \tag{39.9}$$

(39b) The integrated curvature. In any curved geometry, areas are computed by

$$\mathrm{area}(U) = \int_U e^{2\psi(x,y)}\,dx\,dy. \tag{39.10}$$

Similarly, the geometrically correct way to integrate a function $f(x, y)$ over a region U is

$$\int_U e^{2\psi(x,y)} f(x,y)\,dx\,dy. \tag{39.11}$$

This encodes the sense of integral as an average, where regions with larger ψ should count more. Applying that idea to curvature, we define the integrated curvature over a region U to be

$$\int_U e^{2\psi} K\; dx\,dy = \int_U (-\Delta\psi)\,dx\,dy. \tag{39.12}$$

EXAMPLE 39.4. *Suppose that we have a doubly-periodic geometry,*

$$\psi(x, y) = \psi(x + 1, y) = \psi(x, y + 1). \tag{39.13}$$

Then, the integrated curvature over $U = [0, 1]^2$ is zero. This is easy:

$$\int_0^1\int_0^1 (\partial_x^2\psi)\,dx\,dy = \int_0^1 \big(\partial_x\psi\big)\Big|_{x=0}^{x=1} dy = 0 \tag{39.14}$$

by periodicity, and the same for $\partial_y^2\psi$.

One can think of the example above as a statement about curved geometries on the torus (the torus in question is obtained by identifying opposite sides of U). Since the integrated curvature is zero, it's impossible for such a geometry to have curvature which is everywhere positive (or everywhere negative). This is an instance of a general relationship between curvature and topology.

Let's take a bounded region U, with no holes and with smooth boundary. We parametrize the boundary by a curve $c(t)$ with period $T > 0$, meaning $c(t) = c(t+T)$, going anticlockwise around it. Green's theorem says that

$$\int_U (-\Delta\psi)\; dx\; dy = \int_0^T -(\nabla\psi)_{c(t)} \times c'(t)\; dt. \tag{39.15}$$

You're maybe used to one side of Green's formula being a contour integral. Here, we have spelled out that integral using the parametrization. Suppose that c is a geodesic: from the geodesic equation $\times c'(t)$, which is

$$(c''(t) - \|c'(t)\|^2\, \nabla\psi_{c(t)}) \times c'(t) = 0, \tag{39.16}$$

we get

$$\int_0^T -(\nabla\psi)_{c(t)} \times c'(t)\,dt = \int_0^T \frac{c'(t)\times c''(t)}{\|c'(t)\|^2}\,dt. \tag{39.17}$$

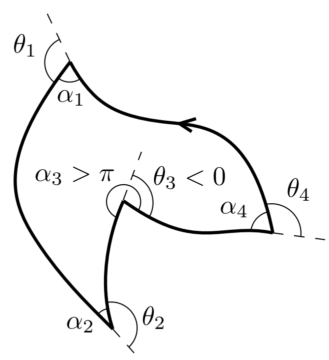
We have seen this integral before, it's just 2π times the rotation number of c. Remember we had Whitney's formula, which relates the rotation number to selfintersections: here we have no selfintersections, and we go anticlockwise, so the formula says that the winding number is $+1$. We put everything together:

THEOREM 39.5. *If the boundary of U is a periodic geodesic, the integrated curvature on U is*

$$\int_U (-\Delta\psi)\,dx\,dy = 2\pi. \tag{39.18}$$

What if instead having a region with smooth boundary, we have one with n corners? The rotation number integral still computes by how much the tangent direction rotates on each side of the boundary, but obviously misses out on the sudden change of tangent direction at the corners. Hence, it computes $2\pi - \theta_1 - \cdots - \theta_n$, where the $\theta_i \in (-\pi, \pi)$ are the angles (positive if counterclockwise, negative if clockwise) by which the tangent direction changes:

(39.19)



In terms of the more familiar interior angles at the corners, $\alpha_k = \pi - \theta_k$, the rotation number integral is $(2-n)\pi + \theta_1 + \cdots + \alpha_n$. The outcome is:

THEOREM 39.6. *(General Gauss-Bonnet) In any curved geometry, let U be a geodesic n-gon. By this we mean a region without holes, whose boundary is a union of n segments of geodesics, coming together at the corners with interior angles $\alpha_k \in (0, 2\pi)$. Then*

$$\int_U (-\Delta\psi)\,dx\,dy = \alpha_1 + \cdots + \alpha_n + (2-n)\pi. \tag{39.20}$$

For hyperbolic geometry, where $K = -1$, the integrated curvature is $-\text{area}(T)$, and this is the theorem we've seen before, in the case of triangles ($n = 3$).