4. The winding number (continued)

In the last lecture, we talked about winding numbers in informal terms. Now,

- we give a trigonometric formula for the winding number. It's not a good tool for computing things by hand, but it puts our discussion on a basis which is independent of intuition.
- We look at some properties and applications of winding numbers.

(4a) A formula for the winding number. Remember the intuition behind the winding number: take a polygonal loop p with vertices $(v_0, v_1, \dots, v_n = v_0)$, and a point q not lying on p. Standing at q, we turn around to follow a point which goes once around p, and count the total number of rotations we have performed. One can do this by measuring the total angle by which we have turned (making sure to count clockwise turning negatively), divided by 2π.

To make this precise, suppose that we have vectors $w_1, w_2 \in \mathbb{R}^2$ which are both nonzero, and which do not point in opposite directions (w_1 is not a negative multiple of w_2). We measure the angle α between those two vectors, and write it as

$$\sphericalangle(w_1, w_2) = \alpha \in (-\pi, \pi); \tag{4.1}$$

it is positive if (w_1, w_2) is a positively oriented basis; negative for negatively oriented bases; and zero if w_1, w_2 are positive multiples of each other. In the application to winding numbers, we are standing and q and watching the segment from v_{k-1} to v_k; the relevant angle is then $\sphericalangle(v_{k-1} - q, v_k - q)$. It will never happen that $v_{k-1} - q = 0$, or that $v_k = 0$, or that $v_{k-1} - q$ and $v_k - q$ point in opposite directions; because any of those would mean that q lies on p. So $\sphericalangle(v_{k-1} - q, v_k - q)$ is always defined. This gives the following formula, which we use as definition of the winding number:

$$\mathrm{wind}(p, q) = \frac{1}{2\pi} \sum_{k=1}^{n} \sphericalangle(v_{k-1} - q, v_k - q). \tag{4.2}$$

(4b) Properties. In the area formula from the last lecture, we saw wind(p, some point $q \in R$), where R was one of the regions into which p divides the plane. This makes sense because of:

Proposition 4.1. *If we move q around without crossing p,* $\mathrm{wind}(p, q)$ *remains constant.*

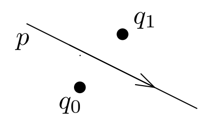
The proof is a classical argument in topology: looking at (4.2) shows that the winding number depends continuously on q (as long as we do not cross p, where the expression becomes ill-defined). But a continuous function can't jump from one integer value to a different one, so $\mathrm{wind}(p, q)$ doesn't change if we move q around (again, unless we cross p).

Corollary 4.2. *Suppose that q can be moved to infinity without crossing p. Then* $\mathrm{wind}(p, q) = 0$.

Because of the previous Proposition, we can assume that q is very very far from p. In this case, the vectors $v_k - q$ are all equal to $-q$ plus an error which is relatively much smaller than q. As a consequence, the angles in (4.2) are very small, and add up to a number which is much smaller in absolute value than 2π, so $|\mathrm{wind}(p,q)| \ll 1$. Since the winding number is an integer, it must be zero!

PROPOSITION 4.3. *Let q_0, q_1 be two points which lie on either side of one of the edges of p, as follows:*

(4.3)



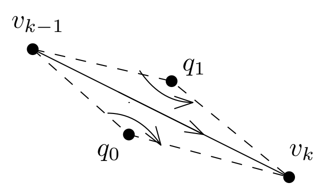
We assume that all other edges lie outside the picture (this is important!). Then $\mathrm{wind}(p,q_1) = \mathrm{wind}(p,q_0) + 1$.

Let's think of q_0, q_1 as lying very close to each other, and that the edge which is being crossed is $\overline{v_{k-1}v_k}$. Its contribution to the winding numbers is

$$\sphericalangle(v_{k-1} - q_0, v_k - q_0) \approx -\pi, \quad \sphericalangle(v_{k-1} - q_1, v_k - q_1) \approx \pi, \tag{4.4}$$

as the following picture shows:

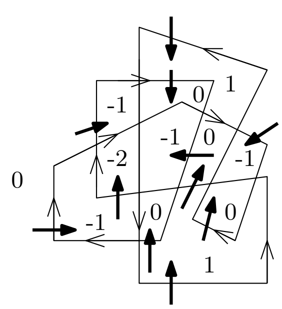
(4.5)



The other edges contribute approximately the same to $\mathrm{wind}(p,q_0)$ and $\mathrm{wind}(p,q_1)$. Therefore, $\mathrm{wind}(p,q_1) \approx \mathrm{wind}(p,q_0) + 1$. But since we are talking about integers, an approximate equality with small error is necessarily a strict equality.

EXAMPLE 4.4. *We compute the winding numbers region by region, starting with the outside (the fat arrows show one possible direction of reasoning):*

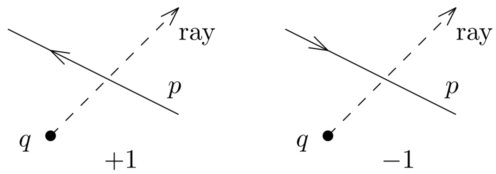
(4.6)



Proposition 4.3 leads to another algorithm for computing winding numbers. Choose a ray (half-line) starting at q and going to infinity, subject to:

(4.7) the ray must avoid the vertices of p, and intersect each edge in at most one point.

Count the intersection points between our ray and the edges of p, with signs:

(4.8) 

Explicitly, our ray is determined by a nonzero vector w, and the sign depends on whether $(w, v_k - v_{k-1})$ is an oriented basis or not. One can write the outcome as:

$$\operatorname{wind}(p,q) = \sum_{\substack{\text{those } 1 \leq k \leq n \text{ for which} \\ \text{the ray intersects } \overline{v_{k-1}v_k}}} \operatorname{sign}(w \times (v_k - v_{k-1})). \tag{4.9}$$

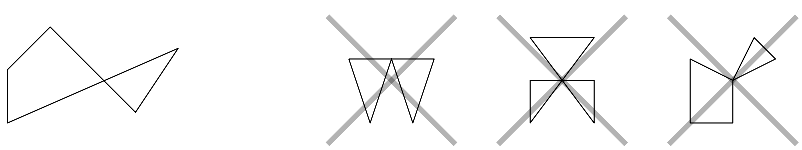
This is the first of several formulae of the same kind that we will encounter: each computes a topological quantity by counting points with ± 1 signs, and in order to work, they require some linear independence condition, in this case (4.7).

Remark 4.5. *The sign conventions in* (4.3) *and* (4.8) *may look like opposites, but are consistent: in one situation, we're computing how the winding number around q_1 differs from that around q_0; in the other, we're computing the winding number at the starting point q of the ray.*

Suppose that P is a polygon. Choose a ray (4.7). Each intersection point of that ray with the edges of P contributes ± 1 to the winding number. The winding number is even (0) if q lies outside P, and odd (± 1, depending on how our numbering of the vertices goes around P) if q lies inside P. This leads to the "point-in-polygon test":

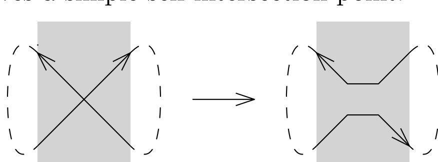
$$q \text{ lies } \begin{cases} \text{outside} \\ \text{inside} \end{cases} P \text{ if the ray intersects the edges of } P \text{ an } \begin{cases} \text{even} \\ \text{odd} \end{cases} \text{ number of times.} \tag{4.10}$$

(4c) A topological application. A simple self-intersection of a polygonal loop is a point where two edges cross: that point is not allowed to be a vertex, and should not lie on any other edge. Here's a loop with a simple self-intersection, and three examples that have more complicated self-intersections (which we don't want here):

(4.11) 

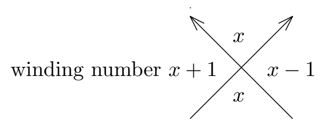
Proposition 4.6. *Take a polygonal loop which has N simple self-intersections, and no self-intersections of any other kind. divides the plane into $N+2$ regions.*

There is a process that removes a simple self-intersection point:

(4.12) 

It's important to do it as indicated, so that the outcome is a single loop, and not two of them! Winding numbers show that in the picture above, the parts to the left and right of the intersection point belong to different regions:

(4.13)



After we remove the selfintersection point, those two regions get merged. Therefore, (4.12) decreases the number of regions by 1. Repeated application reduces the statement of Proposition 4.6 to the familiar case $N = 0$ of polygons.

Example 4.7. *Here's a repeated application of that process, together with the winding numbers:*

(4.14)