40. Geometry of combinatorial surfaces

A priori, combinatorial surfaces seem hardly the kind of object that can have curvature, but one can in fact make sense of that. The triangles makinf up the surface are thought of as flat, and the curvature is concentrated at the vertices (like a Dirac δ-function), where it appears in the form of an angle defect. In this lecture,

- we introduce combinatorial surfaces with added geometric data;
- we explain a version of Gauss-Bonnet for such surfaces;
- finally, we return to our discussion of polygonal billiards, and explain how this gives rise to a particular construction of surfaces.

(40a) Geometric combinatorial surfaces. Recall that a combinatorial surface S is a special case of an abstract complex. As such, it is constructed from vertices, edges and triangles, according to given incidence rules. A *geometry* on a combinatorial surface is specified like this: each edge (i,j) should be given a positive length $l_{ij} > 0$; such that for each triangle (p,q,r), we have the triangle inequalities

$$\begin{aligned} l_{pr} &< l_{pq} + l_{qr}, \\ l_{pq} &< l_{pr} + l_{qr}, \\ l_{qr} &< l_{pq} + l_{pr}. \end{aligned} \tag{40.1}$$

Visually, one thinks of each triangle as drawn in the Euclidean plane (up to Euclidean transformations, which means that only its congruence class matters), in a way which is compatible with the prescribed edge-lengths. This realizability is equivalent to (40.1), and the lengths determine the congruence class.

Look at a vertex of our surface. By definition, it is surrounded by a collection of triangles, which can be drawn in the plane as in (31.1). However, this picture isn't necessarily compatible with realizing the triangles geometrically; in the relevant congruence classes, the angles surrounding the vertex may not add up to 2π. We give this defect a name: the *discrete curvature* at our vertex is

$$\kappa_{vertex} = 2\pi - \sum(\textit{adjacent angles of the triangles at the vertex}). \tag{40.2}$$

For bookkeeping purposes, since the vertices are numbered as $1,\dots,n_0$, we have discrete curvatures $\kappa_1,\dots,\kappa_{n_0}$.

Proposition 40.1. *The sum $\sum_{i=1}^{n_0} \kappa_i$ of the discrete curvatures at all vertices of the surface S equals $2\pi\chi(S)$, where χ is the Euler characteristic.*

This is completely elementary: $\sum_i \kappa_i$ is $2\pi n_0$ (the number of vertices) minus the sum of all angles that occur on our surface, which is πn_2 (the number of triangles). Because of the surface condition, the number of edges is $n_1 = \frac{3}{2}n_2$. Therefore,

$$2\pi\chi(S) = 2\pi(n_0 - n_1 + n_2) = 2\pi(n_0 - n_2/2) = 2\pi n_0 - \pi n_2 = \sum_i \kappa_i. \tag{40.3}$$

Example 40.2. *Let's think of an icosahedron, in its original incarnation as a Platonic solid (all edges have the same length). Since there are five equilateral triangles adjacent to each vertex,* $\kappa_i = 2\pi - 5(\pi/3) = \pi/3$. *There are twelve vertices, so* $\sum_i \kappa_i = 4\pi$. *Indeed, we know that the Euler characteristic is* 2.

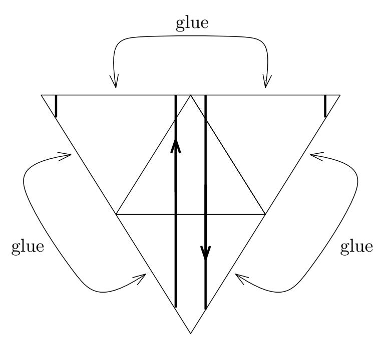
Example 40.3. *For a torus, realized in any way as a combinatorial surface, the Euler characteristic is zero. Hence, the discrete curvatures must sum to zero (either they are all zero, or else curvatures of either sign must occur).*

Example 40.3 should remind you of the statement about the integral of the Gaussian curvature for a doubly periodic geometry, Example 39.4. Indeed, there is a Gauss-Bonnet theorem for curved surfaces, which we can't state, not having the definition of such a surface; and then, Proposition 40.1 can be viewed as the discrete combinatorial analogue of that theorem.

(40b) Geodesics on combinatorial surfaces. A geodesic on the ordinary plane is just a straight line. Similarly, in a combinatorial surface that's been equipped with a geometry, we can move inside each triangle in a straight-line constant-speed motion; and whenever we cross an edge, we continue onto the adjacent triangle with the same speed and at the same angle from their common edge. The outcome is called a combinatorial geodesic. If we happen to run into a vertex, the behaviour becomes undefined, and the geodesic will end there.

Example 40.4. *The following picture shows a tetrahedron as one would make it out of paper, folding the outer triangles up and gluing their sides together; and a periodic geodesic on it:*

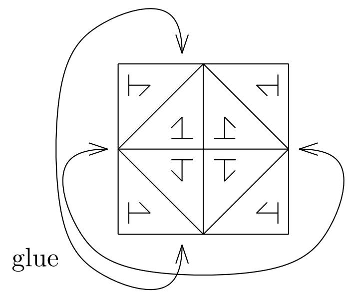
(40.4)



(40c) Translation surfaces. The description of combinatorial geodesics may have reminded you of something, namely our original discussion of polygonal billiards. There is indeed a connection between the two, for polygons with rational angles. Let's more specifically take a triangle whose angles are integer multiples of $180°/N$ for some N. We can associate to such a triangle a combinatorial surface, called its translation surface. To do that, simply take the triangle, and start repeatedly reflecting it along its sides. The reflected triangles will usually start to overlap, but whenever that happens, we avoid it by translating one of the triangles to somewhere else in the plane. Also, we don't want to keep copies which are just translations of each other, so after finitely many reflections, we will have a complete collection. Moreover, whenever we separate a triangle and its reflected copy by translating them apart, let's remember their original common edge (the edge of the reflection). Those edges are glued back together (in an abstract sense, meaning they are the same edge in the abstract complex) to form the *translation surface* associated to our triangle.

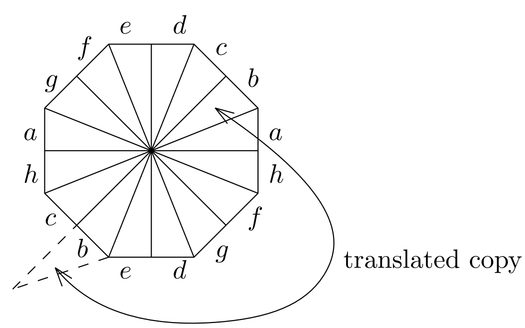
EXAMPLE 40.5. *Take a triangle with angles* $(\pi/4, \pi/4, \pi/2)$. *The associated translation surface is a torus, which consists of the following pattern of triangles with the sides identified.*

(40.5)



Since this triangle has symmetries, we have drawn a symbol inside the triangles so that you can see how they are reflected copies of each other. This is the infinite periodic tiling (6.9) *"rolled up" into a torus. Note that this translation surface has* 4 *vertices: both the center and the corner of the square in* (40.5) *come from the right angle in the original triangle.*

EXAMPLE 40.6. *Take a triangle with angles* $(\pi/8,\ 3\pi/8,\ \pi/2)$. *Composing two reflections on axes that intersect at a* $\pi/8$ *angle gives a* $\pi/4$ *rotation, which we can also do repeatedly; from the other angles we don't get any additional rotations. So, we have* 8 *rotations that we will apply to our triangle, plus reflected versions, making* 16 *triangles. If we first do the reflections around the* $\pi/8$ *vertex, we can draw the* 16 *without overlap:*



(40.6)

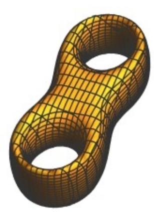
However, other reflections will cause opposite sides of this octagon to become identified with each other (the arrow in our picture indicates how that happens). This identification is not the same as when constructing the projective plane: opposite sides are identified by a translation, not by a $180°$ *rotation!*

Attentive readers will have noticed a problem, clearly visible in (40.5). Namely, a translation surface may violate one of our original conditions in the definition of abstract complex: there can be two different edges that connect the same endpoints. This is just a bookkeeping problem: originally, we had labeled the edges by their pairs of endpoints; now, we'll just have to index them in some other way, and then remember what the endpoints of each edge are. It's like changing the data structures in our computer code; the actual mathematics remains the same, including notions of Euler characteristics, Betti numbers, and orientability. Speaking of the latter, translation surfaces are always orientable (this comes from their construction by reflections).

As one can see from the construction, translation surfaces are naturally geometric. Each vertex of a translation surface comes from one of the original vertices of the triangle. Basically, we keep reflecting along lines passing through that vertex until we get back in original position. This means:

FACT 40.7. *If a vertex of the translation surface comes from a vertex of the triangle with an angle* $\pi\frac{a}{b}$ *(with* a *and* b *coprime), the discrete curvature at our vertex will be* $2\pi(1-a)$*.*

EXAMPLE 40.8. *For the* $(\pi/8, 3\pi/8, \pi/2)$ *triangle, two kinds of vertices (those for the smallest and largest angle) will have discrete curvature zero, but the remaining kind has curvature* -4π*. There is only one such vertex (it corresponds to the vertices of the octagon in* (40.6)*; the identification of opposite sides causes any two of those to become the same). The discrete Gauss-Bonnet theorem says that the Euler characteristic must be* -2*. This is not one of the surfaces we've seen before, it's an "orientable genus two surface". Such surfaces are usually shown in space like this:*



(40.7)

Our original analysis of billiards trajectories, by drawing them as straight lines continuing into reflected triangles, now turns into the following:

FACT 40.9. *The billiards motion in the triangle can be viewed as the motion along geodesics on the associated translation surface.*