5. Loops avoiding two points

Fix two points in the plane, and look at polygonal loops that avoid both points. The possible behaviours of such loops are unexpectedly complicated. How can one encode that complexity?

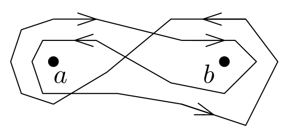
- We introduce a "language" written with four letters, and additional rules for words in that language. This may look strange, but it's actually easy to work with.
- Each loop gives rise to a "word" in our language. One can read off the winding numbers from the word, but it contains much more information than they do.

(5a) Loops that avoid two points. One can think of the winding number $\mathrm{wind}(p, q)$ as describing the topology of polygonal loops p which avoid q. Different winding numbers correspond to qualitatively different behaviours. What if we fix two points a and b, and look at loops that avoid both? There are now two winding numbers $\mathrm{wind}(p, a)$ and $\mathrm{wind}(p, b)$, but one gets the feeling that this does not describe the situation completely. Here are two loops with $\mathrm{wind}(p, a) = 2$, $\mathrm{wind}(p, b) = 0$, but which in an intuitive sense behave differently:

(5.1)

and here is a loops with both winding numbers zero, but which is still somehow nontrivial:

(5.2)



There is a more sophisticated topological invariant which encodes such complexity. However, it is not a number!

(5b) Letters and words. Take a language which is written using only four letters: A, A^{-1} ("A-inverse"), B, B^{-1} ("B-inverse"). A word is an arbitrary sequence of such letters, put inside square brackets to remind us that it's part of our language game:

(5.3) $$[A],\ [AABA],\ [BAB^{-1}A^{-1}], \dots \text{ as well as the empty (trivial) word } [].$$

There are two rules concerning words. First, when a pair of letters AA^{-1}, $A^{-1}A$, BB^{-1}, $B^{-1}B$ occurs, we can cancel it. So

(5.4) $$[BB^{-1}ABA^{-1}A] = [AB]. \quad [A^{-1}BAA^{-1}B^{-1}A] = [].$$

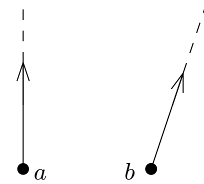
In reverse, one can also insert a cancelling pair anywhere into a word, if one wants to. We regard this as still being the same word. Second, you can move a letter from the start to the end of the word, or from the end to the start. You can also do this several times:

(5.5) $$[BABA^{-1}B] = [ABA^{-1}BB] = [BA^{-1}BBA].$$

Again, all these are thought of as being the same word. Even though you can move a letter from the start to the end and back, you can't move letters around arbitrarily: $[ABAB]$ and $[AABB]$ are different words.

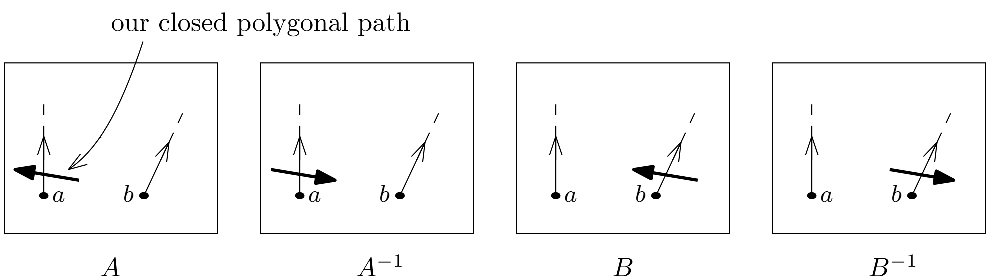
(5c) From loops to words. Let p be a polygonal loop which avoids both a and b. Send out a ray from each of our two points a and b to infinity, so that the rays don't intersect each other:

(5.6)



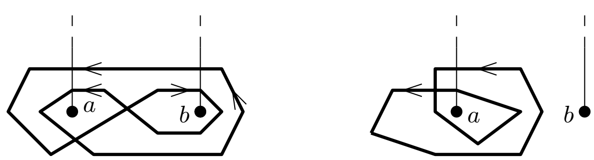
Those rays should be chosen so that they don't meet the vertices of p, and intersect each edge of p in at most one point, just as in (4.7) from the last lecture. We go once around p and write down a word left-to-right. Each letter corresponds to an intersection point with a ray, and they are assigned following this rule:

(5.7)



For instance, the words associated to the loops from (5.1) are

(5.8)



$[BAB^{-1}A]$ $[AA]$

By comparing the instructions above with those for computing winding numbers, one sees:

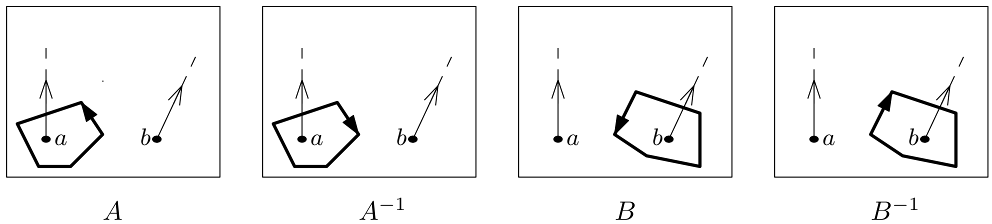
Fact 5.1. *The winding numbers of a loop can be read off from the associated word:* $\mathrm{wind}(p, a)$ *is the number of A letters minus the number of A^{-1} letters; and* $\mathrm{wind}(p, b)$*, the number of B letters minus the number of B^{-1} letters.*

However, our language cares not just about counting intersection points with the rays, but also about the order in which those points appear along the loop. For instance, the loop from (5.2) yields the word $[BA^{-1}B^{-1}A]$, which is not the empty word $[]$.

Fact 5.2. *Every word in our language comes from some loop.*

Here's the proof: given a word, we can construct a corresponding loop by taking basic pieces for the letters, which are all loop with the same starting point, and going around one, then a second one, and so on, as required:

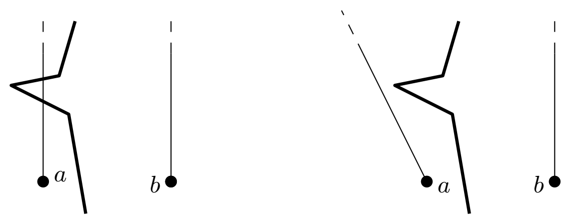
(5.9)



THEOREM 5.3. *The word associated to p is independent of the choice of rays. It also remains the same if we move a and b (as long as we don't cross p).*

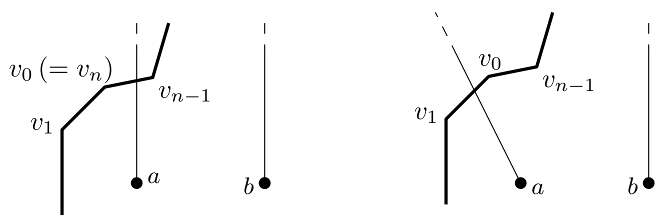
The reason is this: moving the ray, or the points, can lead to the appearance or disappearance of an AA^{-1}, and similarly for other cancelling pairs.

(5.10)



It can also lead to moving the last letter to the first position (and vice versa),

(5.11)



With some effort, one can show that this takes into account anything that can happens when moving points and rays.

(5d) Topological implications. Let's see how words interact with geometric properties of polygonal loops.

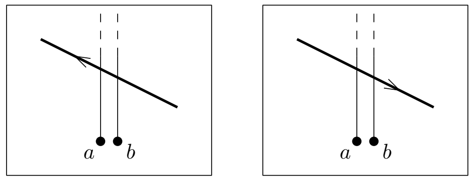
PROPOSITION 5.4. *Suppose that we can move a to infinity without crossing p. Then the word of p is one of the following: [], $[B \cdots B]$ (a bunch of repeated B) or $[B^{-1} \cdots B^{-1}]$ (a bunch of repeated B^{-1}).*

This is a consequence of Theorem 5.3, by contradiction: if one could move a far away, the ray emanating from it could be chosen not to intersect p at all. This means that we get a word involving only the letters B and B^{-1}. But since those two cancel, we can reduce our word until only B or only B^{-1}, or the empty word, remain.

PROPOSITION 5.5. *Suppose that we can move from a to b without crossing p. Then the word of p is one of the following: [], $[BA \dots BA]$ (an even number of letters, with A and B alternating), $[A^{-1}B^{-1} \dots A^{-1}B^{-1}]$ (an even number of letters, with A^{-1} and B^{-1} alternating).*

The argument is similar: one moves a close to b and chooses the rays parallel and to each other, so that each intersection point with ray is followed by an intersection point with the other. The resulting word is made up out of pieces BA or $A^{-1}B^{-1}$.

(5.12)



BA $\qquad$ $A^{-1}B^{-1}$

BA and $A^{-1}B^{-1}$ cancel each other out, so in the end, only one of the two kinds, or the empty word, is left.

PROPOSITION 5.6. *For an actual polygon, the word is one of these:* [], $[A]$, $[A^{-1}]$, $[B]$, $[B^{-1}]$, $[AB] = [BA]$, $[A^{-1}B^{-1}] = [B^{-1}A^{-1}]$.

The idea is that a polygon can go either clockwise or counterclockwise, and we can place the points a, b either both outside, both inside, or one of each. To turn this into a rigorous argument, one again needs Theorem 5.3: if one of our points is outside the polygon, one can move it far away, and then choose a ray which doesn't intersect the polygon at all; if it's inside, one can move it (for instance) close to the topmost vertex, and then choose a ray which intersects the polygon exactly once.