6. Introduction to billiards

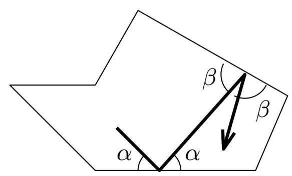
Polygonal billiards (or snooker, or pool, or the classic videogame Pong) is the study of Newtonian motion of a point inside a polygon. The interesting aspect is the long-term behaviour of the trajectories of motion. To study that,

- we introduce the simple idea of drawing mirror copies of our polygon. This is generally helpful; and in very special cases, it explains the billiards behaviour completely.
- Using that idea, we investigate the existence of periodic billiards trajectories.

The contents of this lecture belong to elementary geometry, and don't give a good picture of the intricacy of billiards. We will make up for that in the next lecture, where some theoretical muscle will be brought to bear.

(6a) Playing billiards. Suppose that we have a polygon. Inside it, a pointlike ball is moving in a straight line, bouncing off the edges according to the reflection (equal angle) law. We call the path of its motion a *billiards trajectory*.

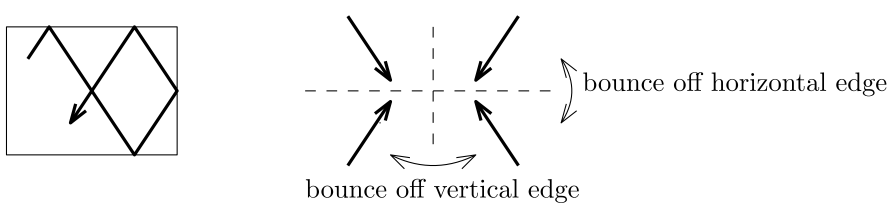
(6.1)



If the ball hits a vertex of the polygon, we declare the behaviour after that to be undefined (for a corner with a general angle, there's no good way to decide how the trajectory should be continued).

If our polygon is a rectangle, any one trajectory only goes in four directions: the direction in which it was originally pointed; the directions obtained from the original one by horizontal or vertical reflection (assuming our rectangle is drawn parallel to the coordinate axes); and the original direction reversed.

(6.2)



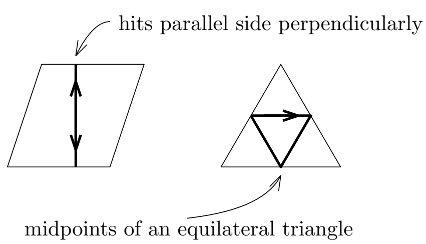
While this is a particularly simple situation, there's a class of polygons to which a similar idea applies. Namely, let's say that P is a *rational-angle polygon* if all interior angles are rational multiples of $360°$ (so, a $44°$ angle would be allowed, but a $(360/\sqrt{2})°$ one would not).

Proposition 6.1. *Suppose that we have a rational-angle polygon, in which all interior angles are integer multiples of* $180°/M$ *for some natural number* M*. Then, any single billiards trajectory moves in at most* $2M$ *different directions (here, directions refers to the vector which gives the velocity).*

For instance, this applies to a rectangle with $M = 2$; and to an equilateral triangle and a regular hexagon, with $M = 3$. To see why the Proposition is true, let's draw a line through the origin parallel to one of the walls, and then all the other lines obtained from it by rotating by multiples of $180°/M$. When bouncing off a wall, the direction of a billiards trajectory is changed by reflection along one of those M lines. Now, if we have two lines which form an angle α, then the composition of the two reflections is a rotation with angle 2α (either clockwise or anticlockwise). For our lines, α is a multiple of $180°/M$, so the rotation is by some multiple of $360°/M$. So, if we follow a trajectory and it bounces off walls 6 times, then the new direction is obtained from the original one by 6 reflections, but we can also think of those as three rotations, where the total angle is still an integer multiple of $360°/M$. If we follow the trajectory through to the 7th bounce, the new direction is given by 7 reflections, or equivalently 1 reflection followed by 3 rotations. Eventually, one sees that all possible directions are obtained from the original one by using either rotations with angle a multiple of $360°/M$, or a reflection corresponding to the first bounce followed by the same kind of rotations. (In contrast, for polygons that do not have the rational-angle property, a single billiards trajectory can go in infinitely many directions, since the different reflections can combine to yield all sorts of angles of rotation.)

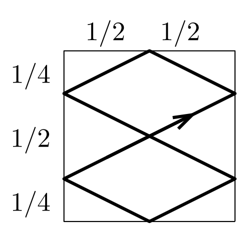
A special kind of billiards trajectories are periodic ones, which repeat the same motion after bouncing off edges a certain number of time. Here are two simple examples, with 2 and 3 bounces:

(6.3)

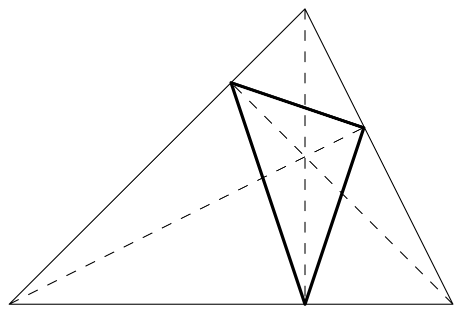


One could go around those trajectories some number N times, and that would be considered a periodic trajectory with $2N$ or $3N$ bounces (but not a particularly interesting one). Here is a 6-bounce periodic trajectory in a square:

(6.4)



Given any acute (all angles $<90^\circ$) triangle, the base points of the altitudes form a smaller triangle, called the orthic (or Fagnano) triangle, which is a 3-bounce trajectory:



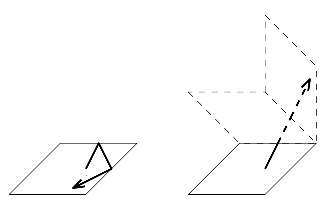
(6.5)

Checking that a specific periodic trajectory works is elementary, since one only needs to verify the incoming-angle-equals-outgoing angle property; at worst, as in the case of the orthic triangle, that turns into an extended exercise in Euclidean geometry. Finding periodic trajectories is an entirely different story. One might think that Proposition 6.1 would help to solve it, but it falls short, since it only regards the direction, and not the position, of a trajectory. Indeed, most trajectories in a square (those with irrational slope) are not periodic. This remains a lively topics in mathematics, for instance the answer to the following is not known:

QUESTION 6.2. *Is there a periodic billiards trajectory in every triangle?*

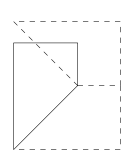
(6b) Mirror polygons. Switching metaphors, let's think of the edges as mirrors, and of our ball as a ray of light. It is a natural idea to add a copy of the polygon that's reflected along one of the edges, as if we were ourselves standing inside the polygon and looking at the mirrors, or as in the image produced by a kaleidoscope. One could then think of a billiards trajectory as continuing straight into the reflected polygon, instead of bouncing off the edge. We call this an *unfolded billiards trajectory.* One can repeat the process, adding more reflected copies:

(6.6)



In general, that runs into problems as the reflected polygons start to overlap.

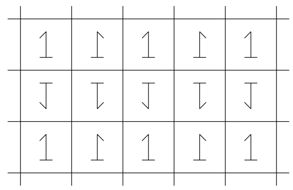
(6.7)



However, there are a few special shapes for which we can continue reflecting infinitely many times, with the resulting copies tiling the plane.

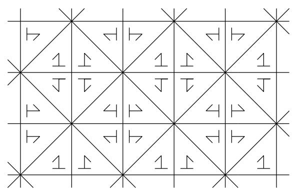
Square or rectangle. The tiling (with a symbol 1 placed on each tile, so that you can see in which way it is a reflected copy) is:

(6.8)



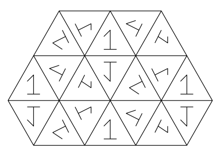
Triangle with angles $45°/45°/90°$:

(6.9)



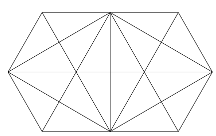
Equilateral triangle:

(6.10)



Triangle with angles $30°/60°/90°$. This has no symmetries, so you can figure out how it's a reflected copy just by looking at the shape, I don't have to mark it with extra stuff:

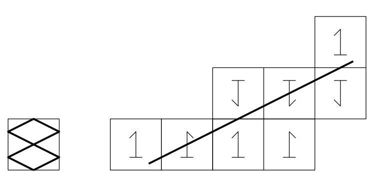
(6.11)



For those very special shapes, one can draw any billiards trajectory in an "unfolded" way, as a straight line passing through the plane tiled with reflected polygons. This makes it easy to find periodic trajectories! Take two tiles which are oriented in the same way (one is the original polygon, and the other a translated copy of it). Pick points on those which correspond to each other, and join them by a straight line. That line, assuming it avoids vertices, is the unfolded version of a periodic billiards trajectory.

Example 6.3. *Here is the* 6*-bounce periodic trajectory in the square, from* (6.4)*, in unfolded and folded form:*

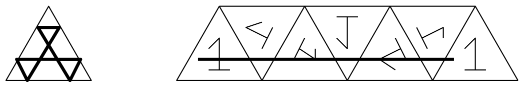
(6.12)



One can translate the line, and that gives an infinite family of periodic trajectories.

EXAMPLE 6.4. *Here is a 6-bounce periodic trajectory in an equilateral triangle:*

(6.13)



Again, there is an infinite family of such trajectories. One of them, where one starts and ends in the middle of one edge, is a 2-fold repeat of the 3-bounce trajectory from (6.3).

(6c) Other polygonal shapes. In principle, one can try to use the reflection trick in other situations as well, but then one has to figure out in each case to what extent it works. As a strategy for finding periodic orbits, it goes as follows:

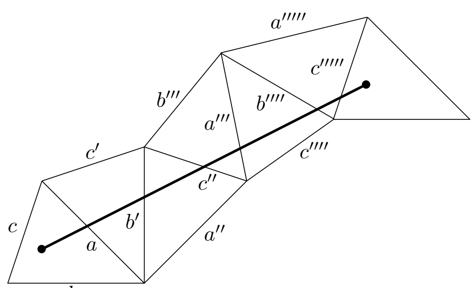
- (Unfolding) Add reflected copies to the original polygon, until we get to one which is a translated version of the original. These copies may not overlap.
- (Finding a trajectory) Draw a straight line segment from a point in the original polygon to its counterpart in the translated copy. This segment must be contained in the union of the non-overlapping polygons we drew, and may not pass through any vertex.
- (Folding back up) By copying the pieces of the straight line segment back into the original polygon, one gets a periodic trajectory, with one bounce for each edge we crossed.

EXAMPLE 6.5. *For the equilateral triangle, we found a* 6*-bounce trajectory in* (6.13)*. Let's look at a triangle which is close to equilateral, and see if we can do the same thing. Write the sides as* (a, b, c)*, and angles as* (α, β, γ)*. Suppose that we reflect along side* a*, which yields a mirror triangle with sides* $(a' = a, c', b')$*. Reflect the new triangle along* b'*, which yields another triangle with sides* $(a'', b'' = b', c'')$*. Continue in that way, doing* 6 *reflections in total, using sides in order abcabc. Thinking of how this affects the way in which the triangles are oriented, one gets this simplified picture:*

(6.14)
$$\begin{tikzcd} {} \arrow[r, "a\ \textit{reflection}"] \arrow[rr, bend right, "2\gamma\ \textit{rotation}"'] & {} \arrow[r, "b'\ \textit{reflection}"] & {} \arrow[r, "c''\ \textit{reflection}"] \arrow[rr, bend right, "2\beta\ \textit{rotation}"'] & {} \arrow[r, "a'''\ \textit{reflection}"] & {} \arrow[r, "b''''\ \textit{reflection}"] \arrow[rr, bend right, "2\alpha\ \textit{rotation}"'] & {} \arrow[r, "c'''''\ \textit{reflection}"] & {} \end{tikzcd}$$

The total angle of rotation is $2\gamma + 2\beta + 2\alpha = 360°$*, which means that the final triangle is a translated (not rotated) copy of the original one. One can then get a* 6*-bounce trajectory as follows:*

(6.15)



Of course, to fully justify this, we would have to explain why the 7 *triangles in the picture don't overlap, and why we can find a straight line segment that only goes through those particular* *reflected copies. All that has to do with how close the picture is to the original one for the equilateral triangle: for obtuse triangles the process fails, but for acute ones it can always be made to work, and as a special case one can get the Fagnano trajectory (twice repeated).*

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