9. First computations

The vibrational properties of a piece of string (tied to a fixed position at each end) are familiar: the frequencies are inverse proportional to the length of the string, and they consists of a lowest principal frequency together with its integer multiples (overtones). The two-dimensional situation, where we are looking at a vibrating drum or membrane, is much more interesting, because the geometry of the vibrating region becomes relevant.

- We introduce the resonance frequencies of a region in the plane, with emphasis on the lowest one (the principal frequency).
- We compute the principal frequency for rectangles and equilateral triangles.

(9a) Definition. Take a region U of the plane. It should be of bounded size, meaning not go out to infinity. It should also consist of a single piece, not several disconnected ones. We include the boundary as part of U. This boundary can be straight-lined (polygonal) or curved. Take the Laplace operator, which applied to a function f of (x, y) is

$$\Delta f = \partial_x^2 f + \partial_y^2 f. \tag{9.1}$$

A number $\lambda > 0$ is called a resonance frequency of U if there is a function

$$\begin{cases} f : U \longrightarrow \mathbb{R}, \text{ twice differentiable,} \\ f(x, y) = 0 \text{ for all boundary points of } U\text{; but } f \text{ is not the constant (zero) function,} \\ \Delta f = -\lambda^2 f. \end{cases} \tag{9.2}$$

These f are called resonance modes. The *principal frequency* is the lowest resonance frequency (the full set of frequencies can be quite complicated, unlike the one-dimensional case).

Lemma 9.1. *The resonance frequencies of a region are unchanged under translations, rotations, and reflections (it is a congruence invariant).*

That is pretty straightforward (it's a change of variables that does not affect Δ). Next,

Lemma 9.2. *Scaling up a region (in both directions at once!) by some factor c results in a new region whose resonance frequencies are $1/c$ times those of the original one.*

The definition of the principal frequency as lowest resonance frequency is intuitively nice, but complicated to work with directly. To avoid that, we'll use (but not prove) the following characterization:

Theorem 9.3. *(i) For the principal frequency, there is only one function f satisfying* (9.2)*, up to multiplication by a constant (we call this function the principal mode).*

(ii) Among all resonance frequencies, the principal frequency is the only one for which the function f is either ≥ 0 on all of U, or ≤ 0 on all of U (one can switch signs by multiplying it with a negative constant); any other resonance mode has both positive and negative values.

Part (i) has an interesting consequence: whatever symmetries U might have, are inherited by the principal mode f. Part (ii) is useful because, if we find a function $f \geq 0$ (or $f \leq 0$) which satisfies (9.2) for some λ, then λ must be the principal frequency.

Example 9.4. *Let $U = \{0 \leq x \leq a,\ 0 \leq y \leq b\}$ be a rectangle of size $a \times b$. Because it's an interval in both x and y directions, it makes sense to try to combine the trig functions that one sees in the theory of the one-dimensional vibrating string. With this and the general requirements as motivation, it is not impossible to come up with the function*

$$f(x,y) = \sin(\pi x/a)\sin(\pi y/b), \tag{9.3}$$

which satisfies:

- $f(x,y) = 0$ *on the boundary of the rectangle;*
- $f(x,y) \geq 0$ *everywhere in the rectangle;*
- $\Delta f = -(\pi^2/a^2 + \pi^2/b^2)f$.

The first and third property show that it's a resonance mode. The second property, thanks to (ii) in the previous theorem, shows that's the principal mode. Therefore, the principal frequency of our rectangle is

$$\lambda = \pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}. \tag{9.4}$$

Example 9.5. *Take the equilateral triangle U with side length 1. In coordinates, let's say this is the triangle with vertices $(0,0)$, $(1,0)$, $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$. I pull the following function out of my ass,*

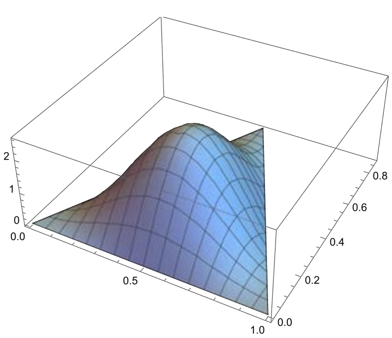
$$f(x,y) = \sin\left(\frac{4\pi}{\sqrt{3}}y\right) + \sin\left(2\pi x - \frac{2\pi}{\sqrt{3}}y\right) + \sin\left(-2\pi x - \frac{2\pi}{\sqrt{3}}y\right). \tag{9.5}$$

This satisfies:

- $f(x,y) = 0$ *zero on the boundary of the triangle;*
- $f(x,y) \geq 0$ *everywhere in the triangle;*
- $\Delta f = -\frac{16}{3}\pi^2 f$.

The first two properties can be seen just by having the computer plot it:

(9.6)



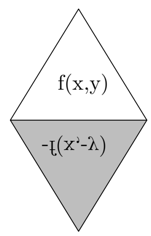
The third one is a differentiation exercise. As before, it follows that we have found the principal mode and hence principal frequency. It is maybe best to scale up the conclusion to any size: the *principal frequency of an equilateral triangle of side-length l is*

$$\lambda = \frac{4}{\sqrt{3}}\pi l^{-1}. \tag{9.7}$$

(9b) The reflection principle. There should be a better way to motivate the function f we've found for the triangle, and in fact there is. This method applies exactly to the shapes whose reflections tile the plane, which we've seen before. Moreover, those are the only shapes where an exact formula for the principal frequency is known!

Let's start with the principal mode f for the equilateral triangle. We enlarge the triangle by reflecting along the x-axis. On the diamond-shape formed by the triangle and its reflection, we consider the function

$$F(x,y) = \begin{cases} f(x,y) & y \geq 0, \\ -f(x,-y) & y \leq 0. \end{cases} \tag{9.8}$$



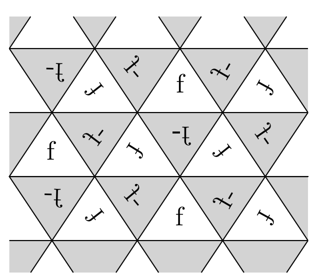
$f(x,-y)$ is what one gets from $f(x,y)$ by applying reflection to the domain of the function. We are doing that and simultaneously changing sign. The change ensures that the function F is nice: continuous, differentiable, and in fact twice differentiable. Think about it like this: on the axis $y = 0$, the derivatives $\partial_x F = 0$ and $\partial_x^2 F = 0$ because the function is zero for all x; the derivatives $\partial_y F$ and $\partial_x \partial_y F$ also exist, as one can see by differentiating $f(x,y)$ and $-f(x,-y)$ (here's where the sign change comes in); and finally, $\partial_y^2 F$ exists because, by the property of f being a principal mode, we have $\partial_y f = -\partial_x^2 f - \lambda^2 f$, and that carries over to $-f(x,-y)$.

One can keep on reflecting until the copies tile the entire plane, you've seen this before! Each time we do that, we carry over the function to the reflected copy and simultaneously change signs, which in the end yields

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}. \tag{9.9}$$

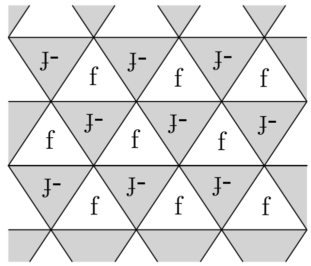
We call this the function obtained by unfolding the original f. The definition of F graphically looks like this, where the orientation of the $\pm f$ reflects how the triangles are reflected copies of the original one, as in (6.10):

(9.10)



Let's clean up our story a little bit. Remember that the principal mode inherits all the symmetries of the original shape. In our case, this means that the function f is invariant under 120° rotations and under reflection that exchange two sides of the triangle. Because of that, we can also just draw the picture like this:

(9.11)



You'll now see many translated copies of the same f. This shows that the function F on the plane has the following periodicity properties:

$$\begin{aligned} F(x+1,y) &= F(x,y), \\ F(x+\tfrac{1}{2}, y+\tfrac{1}{2}\sqrt{3}) &= F(x,y). \end{aligned} \tag{9.12}$$

Of course, alongside those, it also has the property inherited from f,

$$\Delta F = -\lambda^2 F. \tag{9.13}$$

At this point, we turn the argument around: we think generally of functions F with the properties we have just written down, and try to see if we can use that to produce the fundamental mode as the restriction of such a function to the triangle. There is an easy class of trigonometric functions that satisfy (9.13). Namely, given some $v = (c,d) \in \mathbb{R}^2$, define

$$\begin{aligned} S_v &= \sin(2\pi\, v \cdot (x,y)) = \sin(2\pi(cx+dy)), \\ C_v &= \cos(2\pi\, v \cdot (x,y)) = \cos(2\pi(cx+dy)). \end{aligned} \tag{9.14}$$

These satisfy

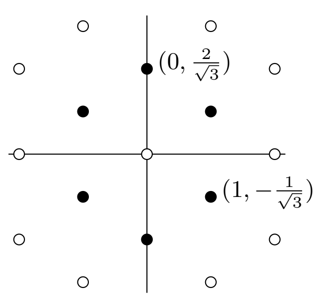
$$\begin{aligned} \Delta S_v &= -4\pi^2\|v\|^2 S_v = -4\pi^2(c^2+d^2)S_v, \\ \Delta C_v &= -4\pi^2\|v\|^2 C_v = -4\pi^2(c^2+d^2)C_v. \end{aligned} \tag{9.15}$$

In order for C_v and S_v to have the same periodicity as in (9.12), we need v to satisfy

$$v \cdot (1,0) \in \mathbb{Z}, \quad v \cdot (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \in \mathbb{Z}. \tag{9.16}$$

The allowed v form a hex grid (not to be confused with the previous pictures!)

(9.17)



We have marked out the 6 black dots which are at distance $\frac{2}{\sqrt{3}}$ from the origin. To these points correspond 6 functions C_v or S_v. All of them satisfy (9.15) with the same $\|v\|^2 = \frac{2}{3}$. Each of those $S_v(x,y)$ separately is zero on one of the sides of the triangle, so to get something that's zero on all three sides, we need to combine them in some way that causes useful cancellations. Here's how to do it:

$$F = S_{(0,\frac{2}{\sqrt{3}})} + S_{(1,-\frac{1}{\sqrt{3}})} + S_{(-1,-\frac{1}{\sqrt{3}})}, \tag{9.18}$$

and that is indeed the function (9.5) we came up with before.

18.900 Geometry and Topology in the Plane
Spring 2023