Normality of Linear Continua

Theorem E.1. Every linear continuum X is normal in the order topology.

Proof. It suffices to consider the case where X has no largest element and no smallest element. For if X has a smallest x_0 but no largest, we can form a new ordered set Y by taking the disjoint union of $(0,1)$ and X, and declaring every element of $(0,1)$ to be less than every element of X. The ordered set Y is a linear continuum with no largest or smallest. Since X is a closed subspace of Y, normality of Y implies normality of X. The other cases are similar.

So suppose X has no largest or smallest. We follow the outline of Exercise 8 of §32.

Step 1. Let C be a nonempty closed subset of X. We show that each component of $X - C$ has the form $(c,+\infty)$ or $(-\infty,c)$ or (c,c'), where c and c' are points of C.

Given a point x of $X - C$, let us take the union U of all open intervals (a_α, b_α) of X that contain x and lie in $X - C$. Then U is connected. We show that U has one of the given forms, and that U is one of the components of $X - C$.

Let $a = \inf a_\alpha$ or $a = -\infty$, according as the set $\{a_\alpha\}$ has a lower bound or not. Let $b = \sup b_\alpha$ or $b = +\infty$ according as the set $\{b_\alpha\}$ has an upper bound or not. Then $U = (a,b)$. If $a \neq -\infty$, we show a is a point of C. Suppose that a is not a point of C. Then there is an open interval (d,e) about a disjoint from C. This open interval contains a_α for some α because $a = \inf a_\alpha$; then the union $(d,e) \cup (a_\alpha, b_\alpha)$ is an open interval that contains x and lies in $X - C$. This contradicts the definition of a.

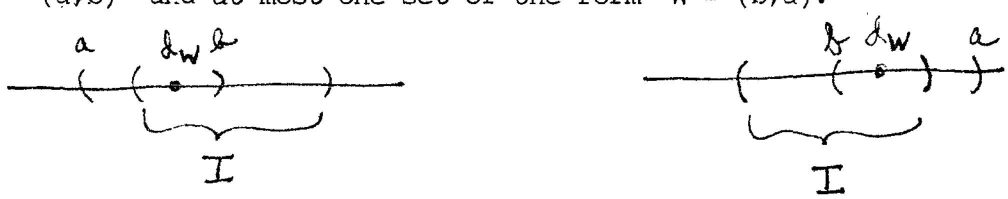
Similarly, if b is not $+\infty$, then b must be a point of C. We conclude that U is of one of the specified forms. [The form $(-\infty,+\infty)$ is not possible, since C is nonempty.]

It now follows that, because the end points of U are $\pm\infty$ or in C, no larger subset of $X - C$ can be connected. Thus U must be the component of $X - C$ that contains x.

Step 2. Let A and B be disjoint closed sets in X. For each component W of $X - A\cup B$ that is an open interval with one end point in A and the other in B, choose a point d_W in W. Let D be the set of all the points d_W. We show that D is closed and discrete.

We show that if x is a limit point of D, then x lies in both A and B (which is not possible). It follows that D has no limit points.

We suppose that x is not in A, and show that x is not a limit point of D. Let I be an open interval about x that is disjoint from A; we show that I contains at most two points of D. If I contains the point d_W of D, then I intersects the corresponding set W, which has one of the forms $W = (a,b)$ or $W = (b,a)$, where $a\in A$ and $b\in B$. Because I is disjoint from A, it can intersect at most one set of the form $W = (a,b)$ and at most one set of the form $W = (b,a)$.



Step 3. Let V be a component of $X - D$. We show that V cannot intersect both A and B.

Suppose V contains a point a of A and a point b of B; assume for convenience that $a<b$. Being connected, V must contain the interval $[a,b]$. Let a_0 be the supremum of the set $A\cap[a,b]$. Then a_0 lies in A and $a_0<b$. The set $(a_0,b]$ does not intersect A. Let b_0 be the infemum of the set $B\cap[a_0,b]$. Then b_0 lies in B and $b_0>a_0$. The interval (a_0,b_0) contains no point of $A\cup B$; because its end points lie in $A\cup B$, no larger subset of $X - A\cup B$ can be connected. Hence (a_0,b_0) is one of the components of $X - A\cup B$; as such, it contains a point of D. Hence V contains a point of D, contrary to construction.

Step 4. By Step 1, the components of $X - D$ are open sets of X. Let U_A be the union of all components of $X - D$ that intersect A, and let U_B be the union of all components of $X - D$ that intersect B. Then U_A and U_B are disjoint open sets containing A and B, respectively. □