3 Categories, functors, natural transformations

From spaces and continuous maps, we constructed graded abelian groups and homomorphisms. We now cast this construction in the more general language of category theory.

Our discussion of category theory will be interspersed throughout the text, introducing new concepts as they are needed. Here we begin by introducing the basic definitions.

Definition 3.1. A *category* $\mathcal{C}$ consists of the following data.

- a class $\mathrm{ob}(\mathcal{C})$ of *objects*;
- for every pair of objects X and Y, a set of *morphisms* $\mathcal{C}(X,Y)$;
- for every object X an *identity morphism* $1_X \in \mathcal{C}(X,X)$; and
- for every triple of objects X,Y,Z, a *composition* map $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$, written $(f,g) \mapsto g \circ f$.

These data are required to satisfy the following:

- $1_Y \circ f = f$, and $f \circ 1_X = f$.
- Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

Note that we allow the collection of objects to be a class. This enables us to talk about a "category of all sets" for example. But we require each $\mathcal{C}(X,Y)$ to be set, and not merely a class. Some interesting categories have a *set* of objects; they are called *small categories*.

We will often write $X \in \mathcal{C}$ to mean $X \in \mathrm{ob}(\mathcal{C})$, and $f\colon X \to Y$ to mean $f \in \mathcal{C}(X,Y)$.

Definition 3.2. If $X, Y \in \mathcal{C}$, then $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon Y \to X$ with $f \circ g = 1_Y$ and $g \circ f = 1_X$. We may write

$$f : X \xrightarrow{\cong} Y$$

to indicate that f is an isomorphism.

Example 3.3. Many common mathematical structures can be arranged in categories.

- Sets and functions between them form a category **Set**.
- Abelian groups and homomorphisms form a category **Ab**.
- Topological spaces and continuous maps form a category **Top**.
- Chain complexes and chain maps form a category ch**Ab**.
- A monoid is the same as a category with one object, where the elements of the monoid are the morphisms in the category. It's a small category.
- The sets $[n] = \{0, \dots, n\}$ for $n \geq 0$ together with weakly order-preserving maps between them form the *simplex category* $\mathbf{\Delta}$, another small category. It contains as a subcategory the *semi-simplex category* $\mathbf{\Delta}_{inj}$ with the same objects but only injective weakly order-preserving maps.
- A partially ordered set or "poset" forms a category in which there is a morphism from x to y iff $x \leq y$. A small category is a poset exactly when (1) there is at most one morphism between any two objects, and (2) the only isomorphisms are identities. This is to be distinguished from the category of posets and order-preserving maps between them, which is "large."

Categories may be related to each other by rules describing effect on both objects and morphisms.

Definition 3.4. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of the data of

- an assignment $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, and
- for all $X, Y \in \mathrm{ob}(\mathcal{C})$, a function $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$.

These data are required to satisfy the following two properties:

- For all $X \in \mathrm{ob}(\mathcal{C})$, $F(1_X) = 1_{F(X)} \in \mathcal{D}(F(X), F(X))$, and
- For all composable pairs of morphisms f, g in $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$.

We have defined quite a few functors already:

$$\mathrm{Sin}_n : \mathbf{Top} \to \mathbf{Set}\,, \quad S_n : \mathbf{Top} \to \mathbf{Ab}\,, \quad H_n : \mathbf{Top} \to \mathbf{Ab}\,,$$

for example. We also have defined, for each X, a morphism $d : S_n(X) \to S_{n-1}(X)$. This is a "morphism between functors." This property is captured by another definition.

Definition 3.5. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* or *natural map* $\theta\colon F \to G$ consists of maps $\theta_X\colon F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$ such that for all $f\colon X \to Y$ the following diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

So for example the boundary map $d\colon S_n \to S_{n-1}$ is a natural transformation.

Example 3.6. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two categories, and assume that $\mathcal{C}$ is small. We may then form the *category of functors* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Its objects are the functors from $\mathcal{C}$ to $\mathcal{D}$, and given two functors F, G, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})(F, G)$ is the set of natural transformations from F to G. We let the reader define the rest of the structure of this category, and check the axioms. We needed to assume that $\mathcal{C}$ is small in order to guarantee that there is no more than a set of natural transformations between functors.

For example, let G be a group (or a monoid) viewed as a one-object category. An object $F \in \mathrm{Fun}(G, \mathbf{Ab})$ is simply a group action of G on $F(*) = A$, i.e., a representation of G in abelian groups. Given another $F' \in \mathrm{Fun}(G, \mathbf{Ab})$ with $F'(*) = A'$, a natural transformation from $F \to F'$ is precisely a G-equivariant homomorphism $A \to A'$.