## 24. UNIVERSAL COEFFICIENT THEOREM

## 24 Universal coefficient theorem

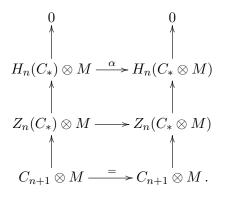
Suppose that we are given  $H_*(X; \mathbf{Z})$ . Can we compute  $H_*(X; \mathbf{Z}/2\mathbf{Z})$ ? This is non-obvious. Consider the map  $\mathbf{RP}^2 \to S^2$  that pinches  $\mathbf{RP}^1$  to a point. Now  $H_2(\mathbf{RP}^2; \mathbf{Z}) = 0$ , so in  $H_2$  this map is zero. But in  $\mathbf{Z}/2\mathbf{Z}$ -coefficients, in dimension 2, this map gives an isomorphism. This shows that there's no *functorial* determination of  $H_*(X; \mathbf{Z}/2)$  in terms of  $H_*(X; \mathbf{Z})$ ; the effect of a map in integral homology does not determine its effect in mod 2 homology. So how *do* we go between different coefficients?

Let R be a commutative ring and M an R-module, and suppose we have a chain complex  $C_*$  of R-modules. It could be the singular complex of a space, but it doesn't have to be. Let's compare  $H_n(C_*) \otimes M$  with  $H_n(C_* \otimes M)$ . (Here and below we'll just write  $\otimes$  for  $\otimes_R$ .) The latter thing gives homology with coefficients in M. How can we compare these two? Let's investigate, and build up conditions on R and  $C_*$  as we go along.

First, there's a natural map

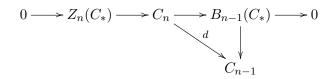
$$\alpha: H_n(C_*) \otimes M \to H_n(C_* \otimes M),$$

sending  $[z] \otimes m$  to  $[z \otimes m]$ . We propose to find conditions under which it is injective. The map  $\alpha$  fits into a commutative diagram with exact columns like this:



Now,  $Z_n(C_* \otimes M)$  is a submodule of  $C_n \otimes M$ , but the map  $Z_n(C) \otimes M \to C_n \otimes M$  need not be injective ... unless we impose more restrictions. If we can guarantee that it is, then a diagram chase shows that  $\alpha$  is a monomorphism.

So let's assume that R is a PID and that  $C_n$  is a free R-module for all n. Then the submodule  $B_{n-1}(C_*) \subseteq C_{n-1}$  is again free, so the short exact sequence



splits. So  $Z_n(C_*) \to C_n$  is a split monomorphism, and hence  $Z_n(C_*) \otimes M \to C_n \otimes M$  is too.

In fact, a little thought shows that this argument produces a splitting of the map  $\alpha$ .

Now,  $\alpha$  is not always an isomorphism. But it certainly is if M = R, and it's compatible with direct sums, so it certainly is if M is free. The idea is now to resolve M by frees, and see where that idea takes us.

So let

$$0 \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M. Again, we're using the assumption that R is a PID, to guarantee that  $\ker(F_0 \to M)$  is free. Again using the assumption that each  $C_n$  is free, we get a short exact sequence of chain complexes

$$0 \to C_* \otimes F_1 \to C_* \otimes F_0 \to C_* \otimes M \to 0$$
.

In homology, this gives a long exact sequence. Unsplicing it gives the left-hand column in the

following diagram.

The right hand column occurs because  $\alpha$  is an isomorphism when the module involved is free. But

$$\operatorname{coker}(H_n(C_*) \otimes F_1 \to H_n(C_*) \otimes F_0)) = H_n(C_*) \otimes M$$

and

$$\ker(H_{n-1}(C_*)\otimes F_1\to H_{n-1}(C_*)\otimes F_0)=\operatorname{Tor}_1^R(H_{n-1}(C_*),M).$$

We have proved the following theorem.

**Theorem 24.1** (Universal Coefficient Theorem). Let R be a PID and  $C_*$  a chain complex of R-modules such that  $C_n$  is free for all n. Then there is a natural short exact sequence of R-modules

$$0 \to H_n(C_*) \otimes M \xrightarrow{\alpha} H_n(C_* \otimes M) \xrightarrow{\partial} \operatorname{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

that splits (but not naturally).

**Example 24.2.** The pinch map  $\mathbb{RP}^2 \to S^2$  induces the following map of universal coefficient short exact sequences:

This shows that the splitting of the universal coefficient short exact sequence cannot be made natural, and it explains the mystery that we began with.

**Exercise 24.3.** The hypotheses are essential. Construct two counterexamples: one with  $R = \mathbf{Z}$  but in which the groups in the chain complex are not free, and one in which  $R = k[d]/d^2$  and the modules in  $C_*$  are free over R.

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