$\square$

13 Proof of the Locality Principle

We have constructed the subdivision operator $\$: S_*(X) \to S_*(X)$, with the idea that it will shrink chains and by iteration eventually render any chain $\mathcal{A}$-small. Does \$ succeed in making simplices smaller? Let's look first at the affine case. Recall that the "diameter" of a subset X of a metric space is given by

$$\mathrm{diam}(X) = \sup\{d(x,y) : x, y \in X\}\,.$$

Lemma 13.1. *Let σ be an affine n-simplex, and τ a simplex in $\$\sigma$. Then* $\mathrm{diam}(\tau) \leq \frac{n}{n+1}\mathrm{diam}(\sigma)$.

Proof. Suppose that the vertices of σ are $v_0, v_1, \ldots, v_n$. Let b be the barycenter of σ, and write the vertices of τ as $w_0 = b, w_1, \ldots, w_n$. We want to estimate $|w_i - w_j|$. First, compute

$$|b - v_i| = \left|\frac{v_0 + \cdots + v_n - (n+1)v_i}{n+1}\right| = \left|\frac{(v_0 - v_i) + (v_1 - v_i) + \cdots + (v_n - v_i)}{n+1}\right|\,.$$

One of the terms in the numerator is zero, so we can continue:

$$|b - v_i| \leq \frac{n}{n+1}\max_{i,j}|v_i - v_j| = \frac{n}{n+1}\mathrm{diam}(\sigma)$$

Since $w_i \in \sigma$,

$$|b - w_i| \leq \max_i |b - v_i| \leq \frac{n}{n+1}\mathrm{diam}(\sigma)\,.$$

For the other cases, we use induction:

$$|w_i - w_j| \leq \mathrm{diam}(\text{simplex in } \$d\sigma) \leq \frac{n-1}{n}\mathrm{diam}(d\sigma) \leq \frac{n}{n+1}\mathrm{diam}(\sigma)\,.$$

□

Now let's transfer this calculation to singular simplices in a space X equipped with a cover $\mathcal{A}$.

Lemma 13.2. *For any singular chain c, some iterate of the subdivision operator sends c to an $\mathcal{A}$-small chain.*

Proof. We may assume that c is a single simplex $\sigma : \Delta^n \to X$, because in general you just take the largest of the iterates of \$ needed to send the simplices in c to a $\mathcal{A}$-small chains. We now encounter another of the great virtues of singular homology: we pull $\mathcal{A}$ back to a cover of the standard simplex. Define an open cover of Δ^n by

$$\mathfrak{U} := \{\sigma^{-1}(\mathrm{Int}(A)) : A \in \mathcal{A}\}\,.$$

The space Δ^n is a compact metric space, and so is subject to the Lebesgue covering lemma, which we apply to the open cover $\mathfrak{U}$.

Lemma 13.3 (Lebesgue covering lemma)**.** *Let M be a compact metric space, and let $\mathfrak{U}$ be an open cover. Then there is $\epsilon > 0$ such that for all $x \in M$, $B_\epsilon(x) \subseteq U$ for some $U \in \mathfrak{U}$.*

□

To apply this, we will have to understand iterates of the subdivision operator.

Lemma 13.4. *For any $k \geq 1$, $\$^k \simeq 1 : S_*(X) \to S_*(X)$.*

Proof. We construct T_k such that $dT_k + T_k d = \$^k - 1$. To begin, we take $T_1 = T$, since $dT + Td = \$ - 1$. Let's apply \$ to this equation. We get $\$dT + \$Td = \$^2 - \$$. Sum up these two equations to get

$$dT + Td + \$dT + \$Td = \$^2 - 1\,,$$

which simplifies to

$$d(\$ + 1)T + (\$ + 1)Td = \$^2 - 1$$

since $\$d = d\$$.

So define $T_2 = (\$ + 1)T$. Continuing, you see that we can define

$$T_k = (\$^{k-1} + \$^{k-2} + \cdots + 1)T\,.$$

□

We are now in position to prove the Locality Principle, which we recall:

Theorem 13.5 (The locality principle)**.** *Let $\mathcal{A}$ be a cover of a space X. The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ is a quasi-isomorphism; that is, $H_*^{\mathcal{A}}(X) \to H_*(X)$ is an isomorphism.*

Proof. To prove surjectivity let c be an n-cycle in X. We want to find an $\mathcal{A}$-small n-cycle that is homologous to c. There's only one thing to do. Pick k such that $\$^k c$ is $\mathcal{A}$-small. This is a cycle because because $\$^k$ is a chain map. I want to compare this new cycle with c. That's what the chain homotopy T_k is designed for:

$$\$^k c - c = dT_k c + T_k dc = dT_k c$$

since c is a cycle. So $\$^k c$ and c are homologous.

Now for injectivity. Suppose c is a cycle in $S_n^{\mathcal{A}}(X)$ such that $c = db$ for some $b \in S_{n+1}(X)$. We want c to be a boundary of an $\mathcal{A}$-small chain. Use the chain homotopy T_k again: Suppose that k is such that $\$^k c$ is $\mathcal{A}$-small. Compute:

$$d\$^k b - c = d(\$^k - 1)b = d(dT_k + T_k d)b = dT_k c$$

so

$$c = d\$^k b - dT_k c = d(\$^k b - T_k c)\,.$$

Now, $\$^k b$ is $\mathcal{A}$-small, by choice of k. Is $T_k c$ also $\mathcal{A}$-small? I claim that it is. Why? It is enough to show that $T_k\sigma$ is $\mathcal{A}$-small if σ is. We know that $\sigma = \sigma_* \iota_n$. Because σ is $\mathcal{A}$-small, we know that $\sigma : \Delta^n \to X$ is the composition $i_*\overline{\sigma}$ where $\overline{\sigma} : \Delta^n \to A$ and $i : A \to X$ is the inclusion of some $A \in \mathcal{A}$. By naturality, then, $T_k\sigma = T_k i_*\overline{\sigma} = i_* T_k \overline{\sigma}$, which certainly is $\mathcal{A}$-small. □

This completes the proof of the Eilenberg Steenrod axioms for singular homology. In the next chapter, we will develop a variety of practical tools, using these axioms to compute the singular homology of many spaces.

Lefschetz progeny

According to the Mathematical Genealogy Project, Solomon Lefschetz had 9312 academic descendents as of March 2018. Here are just a few, with special attention to MIT faculty (marked with an asterisk).