

**Spectral sequences.**

A core confusion in the machinery underlying the theory of spectral sequences has to do with images of iterates of self-maps.

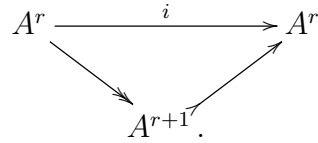
Suppose that  $i : A \rightarrow A$  is an endomorphism of the abelian group  $A$ . Let  $A^r$  denote the group

$$\text{im}(i^{r-1}) = A / \ker(i^{r-1}).$$

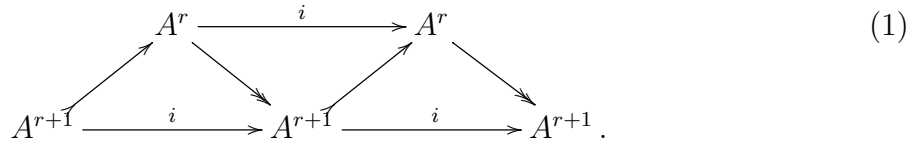
So  $A^1 = A$ . If we regard  $A^r$  as a subgroup of  $A$ , the map  $i$  restricts to a natural map

$$i : A^r \rightarrow A^r.$$

This map factors as



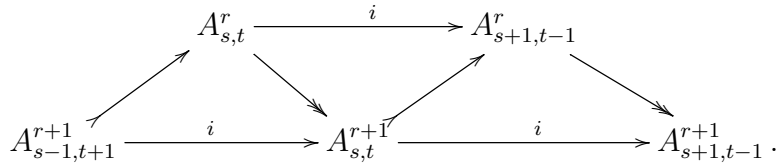
On the other hand,  $i^r a \mapsto i^{r-1}(ia)$  defines an injective map  $A^{r+1} \rightarrow A^r$ , and this map fits into the following commutative diagram.



If we suppose that  $A$  is bigraded and  $\|i\| = (-1, 1)$ , define

$$A_{s,t}^r = \text{im}(i^{r-1} : A_{s,t} \rightarrow A_{s+r-1,t-r+1})$$

so



Now suppose that  $(A^1, i)$  fits into a long exact sequence

$$\cdots \longrightarrow A_{s-1,t+1}^1 \xrightarrow{i} A_{s,t}^1 \xrightarrow{j} E_{s,t}^1 \xrightarrow{k} A_{s-1,t}^1 \xrightarrow{i} A_{s,t-1}^1 \longrightarrow \cdots \quad (2)$$

For example, a filtered complex  $F_*C$  determines a long exact sequence

$$\cdots \longrightarrow H_{s+t}(F_{s-1}C) \longrightarrow H_{s+t}(F_sC) \longrightarrow H_{s+t}(\text{gr}_sC) \longrightarrow H_{s+t-1}(F_{s-1}C) \longrightarrow H_{s+t-1}(F_sC) \longrightarrow \cdots$$

so if we define

$$A_{s,t}^1 = H_{s+t}(F_sC), \quad E_{s,t}^1 = H_{s+t}(\text{gr}_sC).$$

we have an example of (2).

We can embed this row into a big commutative diagram. The squares in the left column come from the right “lozange” in (1), while those in the right column come from the left “lozange.”

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & A_{s-1,t+1}^1 & \xrightarrow{i} & A_{s,t}^1 & \xrightarrow{j} & E_{s,t}^1 & \xrightarrow{k} & A_{s-1,t}^1 & \xrightarrow{i} & A_{s,t-1}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & A_{s-1,t+1}^2 & \longrightarrow & A_{s,t}^2 & & & & A_{s-2,t+1}^2 & \longrightarrow & A_{s-1,t}^2 & & \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & A_{s-1,t+1}^3 & \longrightarrow & A_{s,t}^3 & & & & A_{s-3,t+2}^3 & \longrightarrow & A_{s-2,t+1}^3 & & \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & 
 \end{array}$$

This diagram depends only on the exact sequence (2); but if it arises from a filtered complex, we can interpret  $A_{s,t}^r$  as the  $r$ th approximation to

$$F_s H_{s+t}(C) = \text{im}(H_{s+t}(F_s C) \rightarrow H_{s+t}(C))$$

Certainly we get compatible maps; and if the filtration on  $C$  is exhaustive – that is,

$$C = \cup F_s C = \lim_{s \rightarrow \infty} F_s C$$

– then the filtration on  $H_*(C)$  is also exhaustive, and

$$\lim_{r \rightarrow \infty} A_{s,t}^r = F_s H_{s+t}(C).$$

If we assume furthermore that  $F_{-1}C = 0$ , then  $A_{s,t}^r = 0$  for  $s < 0$ , and the intersection of the right columns in the diagram is zero. So we can adjoin a bottom row to the diagram:

$$\begin{array}{ccccccccccccccc}
 \cdots & \longrightarrow & A_{s-1,t+1}^1 & \xrightarrow{i} & A_{s,t}^1 & \xrightarrow{j} & E_{s,t}^1 & \xrightarrow{k} & A_{s-1,t}^1 & \xrightarrow{i} & A_{s,t-1}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & A_{s-1,t+1}^2 & \longrightarrow & A_{s,t}^2 & & & & A_{s-2,t+1}^2 & \longrightarrow & A_{s-1,t}^2 & & \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & A_{s-1,t+1}^3 & \longrightarrow & A_{s,t}^3 & & & & A_{s-3,t+2}^3 & \longrightarrow & A_{s-2,t+1}^3 & & \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & F_{s-1} H_{s+t}(C) & \longrightarrow & F_s H_{s+t}(C) & \longrightarrow & \text{gr}_s H_{s+t} & \longrightarrow & 0 & & & & 
 \end{array}$$

To get approximations to  $\text{gr}_s H_{s+t}(C)$ , we will now explain how to fill in the rest of the diagram as follows.

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & A_{s-1,t+1}^1 & \xrightarrow{i} & A_{s,t}^1 & \xrightarrow{j} & E_{s,t}^1 & \xrightarrow{k} & A_{s-1,t}^1 & \xrightarrow{i} & A_{s,t-1}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & A_{s-1,t+1}^2 & \longrightarrow & A_{s,t}^2 & \longrightarrow & E_{s,t}^2 & \longrightarrow & A_{s-2,t+1}^2 & \longrightarrow & A_{s-1,t}^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & A_{s-1,t+1}^3 & \longrightarrow & A_{s,t}^3 & \longrightarrow & E_{s,t}^3 & \longrightarrow & A_{s-3,t+2}^3 & \longrightarrow & A_{s-2,t+1}^3 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & 
 \end{array}$$

(Now comes the standard story of derivation of exact couples.)

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