Spectral sequences.

A core confusion in the machinery underlying the theory of spectral sequences has to do with images of iterates of self-maps.

Suppose that $i : A \to A$ is an endomorphism of the abelian group A. Let A^r denote the group

$$\operatorname{im}(i^{r-1}) = A/\ker(i^{r-1})\,.$$

So $A^1 = A$. If we regard A^r as a subgroup of A, the map i restricts to a natural map

$$i : A^r \to A^r\,.$$

This map factors as

$$\begin{tikzcd}
A^r \arrow[rr, "i"] \arrow[dr, two heads] & & A^r \\
& A^{r+1} \arrow[ur, tail] . &
\end{tikzcd}$$

On the other hand, $i^r a \mapsto i^{r-1}(ia)$ defines an injective map $A^{r+1} \to A^r$, and this map fits into the following commutative diagram.

$$\begin{tikzcd}
& A^r \arrow[rr, "i"] \arrow[dr, two heads] & & A^r \arrow[dr, two heads] & \\
A^{r+1} \arrow[ur, tail] \arrow[rr, "i"] & & A^{r+1} \arrow[ur, tail] \arrow[rr, "i"] & & A^{r+1}\,.
\end{tikzcd} \tag{1}$$

If we suppose that A is bigraded and $||i|| = (-1, 1)$, define

$$A^r_{s,t} = \operatorname{im}(i^{r-1} : A_{s,t} \to A_{s+r-1,t-r+1})$$

so

$$\begin{tikzcd}
& A^r_{s,t} \arrow[rr, "i"] \arrow[dr, two heads] & & A^r_{s+1,t-1} \arrow[dr, two heads] & \\
A^{r+1}_{s-1,t+1} \arrow[ur, tail] \arrow[rr, "i"] & & A^{r+1}_{s,t} \arrow[ur, tail] \arrow[rr, "i"] & & A^{r+1}_{s+1,t-1}\,.
\end{tikzcd}$$

Now suppose that (A^1, i) fits into a long exact sequence

$$\cdots \longrightarrow A^1_{s-1,t+1} \xrightarrow{i} A^1_{s,t} \xrightarrow{j} E^1_{s,t} \xrightarrow{k} A^1_{s-1,t} \xrightarrow{i} A^1_{s,t-1} \longrightarrow \cdots \tag{2}$$

For example, a filtered complex F_*C determines a long exact sequence

$$\cdots \longrightarrow H_{s+t}(F_{s-1}C) \longrightarrow H_{s+t}(F_sC) \longrightarrow H_{s+t}(\mathrm{gr}_sC) \longrightarrow H_{s+t-1}(F_{s-1}C) \longrightarrow H_{s+t-1}(F_{s-1}C) \longrightarrow \cdots\,.$$

so if we define

$$A^1_{s,t} = H_{s+t}(F_sC)\,, \quad E^1_{s,t} = H_{s+t}(\mathrm{gr}_sC)\,.$$

we have an example of (2).

We can embed this row into a big commutative diagram. The squares in the left column come from the right "lozange" in (1), while those in the right column come from the left "lozange."

$$
\begin{tikzcd}
\cdots \arrow[r] & A^1_{s-1,t+1} \arrow[r, "i"] \arrow[d, two heads] & A^1_{s,t} \arrow[r, "j"] \arrow[d, two heads] & E^1_{s,t} \arrow[r, "k"] & A^1_{s-1,t} \arrow[r, "i"] & A^1_{s,t-1} \arrow[r] & \cdots \\
& A^2_{s-1,t+1} \arrow[r] \arrow[d, two heads] & A^2_{st} \arrow[d, two heads] & & A^2_{s-2,t+1} \arrow[r] \arrow[u, tail] & A^2_{s-1,t} \arrow[u, tail] & \\
& A^3_{s-1,t+1} \arrow[r] \arrow[d, two heads] & A^3_{st} \arrow[d, two heads] & & A^3_{s-3,t+2} \arrow[r] \arrow[u, tail] & A^3_{s-2,t+1} \arrow[u, tail] & \\
& \vdots & \vdots & & \vdots \arrow[u, tail] & \vdots \arrow[u, tail] &
\end{tikzcd}
$$

This diagram depends only on the exact sequence (2); but if it arises from a filtered complex, we can interpret $A^r_{s,t}$ as the rth approximation to

$$F_sH_{s+t}(C) = \mathrm{im}(H_{s+t}(F_sC) \to H_{s+t}(C))$$

Certainly we get compatible maps; and if the filtration on C is exhaustive – that is,

$$C = \cup F_sC = \lim_{s\to\infty} F_sC$$

– then the filtration on $H_*(C)$ is also exhaustive, and

$$\lim_{r\to\infty} A^r_{s,t} = F_sH_{s+t}(C)\,.$$

If we assume furthermore that $F_{-1}C = 0$, then $A^r_{s,t} = 0$ for $s < 0$, and the intersection of the right columns in the diagram is zero. So we can adjoin a bottom row to the diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & A^1_{s-1,t+1} \arrow[r, "i"] \arrow[d, two heads] & A^1_{s,t} \arrow[r, "j"] \arrow[d, two heads] & E^1_{s,t} \arrow[r, "k"] & A^1_{s-1,t} \arrow[r, "i"] & A^1_{s,t-1} \arrow[r] & \cdots \\
& A^2_{s-1,t+1} \arrow[r] \arrow[d, two heads] & A^2_{st} \arrow[d, two heads] & & A^2_{s-2,t+1} \arrow[r] \arrow[u, tail] & A^2_{s-1,t} \arrow[u, tail] & \\
& A^3_{s-1,t+1} \arrow[r] \arrow[d, two heads] & A^3_{st} \arrow[d, two heads] & & A^3_{s-3,t+2} \arrow[r] \arrow[u, tail] & A^3_{s-2,t+1} \arrow[u, tail] & \\
& \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & & \vdots \arrow[u, tail] & \vdots \arrow[u, tail] & \\
0 \arrow[r] & F_{s-1}H_{s+t}(C) \arrow[r] & F_sH_{s+t}(C) \arrow[r] & \mathrm{gr}_sH_{s+t} \arrow[r] & 0 \arrow[u, tail] & &
\end{tikzcd}
$$

To get approximations to $\mathrm{gr}_s H_{s+t}(C)$, we will now explain how to fill in the rest of the diagram as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & A^1_{s-1,t+1} \arrow[r, "i"] \arrow[d, two heads] & A^1_{s,t} \arrow[r, "j"] \arrow[d, two heads] & E^1_{s,t} \arrow[r, "k"] & A^1_{s-1,t} \arrow[r, "i"] & A^1_{s,t-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & A^2_{s-1,t+1} \arrow[r] \arrow[d, two heads] & A^2_{st} \arrow[r] \arrow[d, two heads] & E^2_{s,t} \arrow[r] & A^2_{s-2,t+1} \arrow[r] \arrow[u, tail] & A^2_{s-1,t} \arrow[r] \arrow[u, tail] & \cdots \\
\cdots \arrow[r] & A^3_{s-1,t+1} \arrow[r] \arrow[d, two heads] & A^3_{st} \arrow[r] \arrow[d, two heads] & E^3_{s,t} \arrow[r] & A^3_{s-3,t+2} \arrow[r] \arrow[u, tail] & A^3_{s-2,t+1} \arrow[r] \arrow[u, tail] & \cdots \\
 & \vdots & \vdots & & \vdots \arrow[u, tail] & \vdots \arrow[u, tail] &
\end{tikzcd}$$

(Now comes the standard story of derivation of exact couples.)