A version of Graeme Segal's perspective on classifying spaces

A group G acting on a set X determines a category GX with object set X and $GX(x,y) = \{g : gx = y\}$. For example $G*$ is the category whose single object has endomorphism monoid G. Its classifying space is denoted BG. The action of G on itself by left translation provides a category GG that is "unicursal" – it's nonempty and has exactly one morphism between any two objects – so its classifying space, denoted EG, is contractible. The group G acts from the left on GG and hence on EG. The map $EG \downarrow BG$ induced by the map of G-sets $G \to *$ is a principal G-bundle. These constructions can also be made internally to the category **Top**. EG is still contractible, and (at least if G is a Lie group) $EG \downarrow BG$ continues to be a principle G-bundle.

A map $\pi : Y \to X$ defines the "descent category" $\check{C}(\pi)$ with objects Y, morphisms $Y \times_X Y$, and structure morphisms the evident maps. There's a natural functor $\epsilon : \check{C}(\pi) \to cX$, where cX denotes the "constant category" with objects and morphisms both given by X. The nerve $N\check{C}(\pi)$ is the "Čech simplicial object" associated to π. It is equipped with a simplicial map to NcX, which is the constant simplicial object with value X.

In particular let $\mathcal{U}$ be a cover of the space X, and let

$$Y = \coprod_{U \in \mathcal{U}} U$$

with its evident projection to X. Write $\check{C}(\mathcal{U})$ in this case, and write $x_{U,V}$ for the morphism determined by $x \in U \cap V$. An n-simplex in $N\check{C}(\mathcal{U})$ is then a sequence of $n+1$ elements of $\mathcal{U}$ together with a point in their intersection.

Suppose we are given a principal G-bundle $p : P \downarrow X$, and a trivializing open cover $\mathcal{U}$ equipped with trivializations (equivariant fiberwise isomorphisms) $t_U : p^{-1}U \to U \times G$.

These data determine a functor

$$\theta : \check{C}(\mathcal{U}) \to G* \ .$$

To decide where to send $x_{U,V}$, look at

$$(U \cap V) \times G \xrightarrow{t_U^{-1}} p^{-1}(U \cap V) \xrightarrow{t_V} (U \cap V) \times G \, .$$

This equivariant map is determined by where $(y, 1)$ is sent, for each $y \in U \cap V$; write $\gamma_{U,V} : U \cap V \to G$ for this map. Then the functor is defined by sending

$$x_{U,V} \mapsto \gamma_{U,V}(x) \, .$$

Passing to classifying spaces, we have

$$\begin{array}{ccc} B\check{C}(\mathcal{U}) & \xrightarrow{\theta} & BG \\ \downarrow{\scriptstyle \epsilon} & & \\ X & & \end{array}$$

Proposition. (1) $\theta^* EG \cong \epsilon^* P$.

(2) If P is numerable, ϵ is a homotopy equivalence.