Chapter 4

Spectral sequences and Serre classes

22 Why spectral sequences?

When we're solving a complicated problem, it's smart to break the problem into smaller pieces, solve them, and then put the pieces back together. Spectral sequences provide a powerful and flexible tool for bridging the "local to global" divide. They contain a lot of information, and can be queried in a variety of ways, so we will spend quite a bit of time getting to know them.

Homology is relatively computable precisely because you can break a space into smaller parts and then use Mayer-Vietoris to put the pieces back together. The long exact homology sequence (along with excision) is doing the same thing. We have seen how useful this is, in our identification of singular homology with the cellular homology of a CW complex. This puts a filtration on a space X, the skeleton filtration, and then makes use of the long exact sequences of the various pairs (X_n, X_{n-1}). Things are particularly simple here, since $H_q(X_n, X_{n-1})$ is nonzero for only one value of q.

There are interesting filtrations that do not have that property. For example, suppose that $p : E \to B$ is a fibration. A CW structure on B determines a filtration of E in which

$$F_s E = p^{-1}(\mathrm{Sk}_s B)\,.$$

Now the situation is more complicated: For each s we get a long exact sequence involving $H_*(F_{s-1}E)$, $H_*(F_s E)$, and $H_*(F_s E, F_{s-1}E)$. The relevant structure of this tangle of long exact sequences is a "spectral sequence." It will describe the exact relationship between the homologies of the fiber, the base, and the total space.

We can get a somewhat better idea of how this might look by thinking of the case of a product projection, $\mathrm{pr}_2 : B \times F \to B$. Then the Künneth theorem is available. Let's assume that we are in the lucky situation in which there is a Künneth isomorphism, so that

$$H_*(B) \otimes H_*(F) \xrightarrow{\cong} H_*(E)\,.$$

You should visualize this tensor product of graded modules by putting the the summand $H_s(B) \otimes H_t(F)$ in degree $n = s + t$ of the graded tensor product in position (s, t) in the first quadrant of the plane. Then the graded tensor product in degree n sums along each "total degree" $n = s + t$. Along the x-axis we see $H_s(B) \otimes H_0(F)$; if F is path connected this is just the homology of the base space. Along the y-axis we see $H_0(B) \otimes H_t(F) = H_t(F)$; if B is path-connected this is just the homology of the fiber. Cross-products of classes of these two types fill out the first quadrant.

The Künneth theorem can't generalize directly to nontrivial fibrations, though, because of examples like the Hopf fibration $S^3 \to S^2$ with fiber S^1. The tensor product picture looks like this

$$\begin{array}{c|cccc} 2 & & & & \\ 1 & \mathbf{Z} & & \mathbf{Z} & \\ 0 & \mathbf{Z} & & \mathbf{Z} & \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

and definitely gives the wrong answer!

What's going on here? We can represent a generating cycle for $H_2(S^2)$ using a relative homeomorphism $\sigma : (\Delta^2, \partial\Delta^2) \to (S^2, o)$. If c_o represents the constant 2-simplex at the basepoint $o \in S^2$, $\sigma - c_o$ is a cycle representing a generator of $H_2(S^2)$. We can lift each of these simplices to simplices in S^3. But a lift of σ sends $\partial\Delta^2$ to one of the fiber circles, and the lift of $\sigma - c_o$ is no longer a cycle. Rather, its boundary is a cycle in the fiber over o, and it represents a generator for $H_1(p^{-1}(o)) \cong H_1(S^1)$.

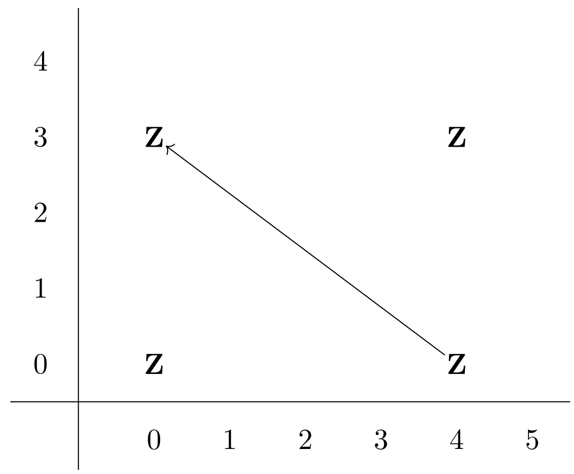
This can be represented by adding an arrow to our picture.

$$\begin{tikzcd} 2 & & & & \\ 1 & \mathbf{Z} & & \mathbf{Z} & \\ 0 & \mathbf{Z} & & \mathbf{Z} \arrow[ull] & \\ & 0 & 1 & 2 & 3 \end{tikzcd}$$

This diagram now reflects several facts: $H_1(S^1)$ maps to zero in $H_1(S^3)$ (because the representing cycle of a generator becomes a boundary!); the image of $H_2(S^3) \to H_2(S^2)$ is trivial (because no nonzero multiple of a generator of $H_2(S^2)$ lifts to a cycle in S^3); and the homology of S^3 is left with just two generators, in dimensions 0 and 3.

In terms of the filtration on the total space S^3, the lifted chain lay in filtration 2 (saying nothing, since $F_2S^3 = S^3$) but not in filtration 1. Its boundary lies two filtration degrees lower, in filtration 0. That is reflected in the differential moving two columns to the left.

The Hopf fibration $S^7 \downarrow S^4$ (which you will study in homework) shows a similar effect. The boundary of the 4-dimensional chain lifting a generating cycle lies again in filtration 0, i.e. on the fiber. This represents a drop of filtration by 4, and is represented by a differential of bidegree $(-4, 3)$.



In every case, the total degree of the differential is of course -1.

The Künneth theorem provides a "first approximation" to the homology of the total space. It's generally too big, but never too small. Cancellation can occur: lifted cycles can have nontrivial boundaries, and cycles that were not boundaries in the fiber can become boundaries in the total space. More complicated cancellation can occur as well, involving the product classes.

Some history

Now I've told you almost the whole story of the Serre spectral sequence. A structure equivalent to a spectral sequence was devised by Jean Leray while he was in a prisoner of war camp during World War II. He discovered an elaborate structure determined in cohomology by a map of spaces. This was much more that just the functorial effect of the map. He was worked in cohomology, and in fact invented a new cohomology theory for the purpose. He restricted himself to locally compact spaces, but on the other hand he allowed *any* continuous map – no restriction to fibrations. This is the "Leray spectral sequence." It's typically developed today in the context of sheaf theory – another local-to-global tool invented by Leray at about the same time.

Leray called his structure an *"anneau spectral"*: he was specifically interested in its multiplicative structure, and he saw an analogy between his analysis of the cohomology of the source of his map and the spectral decomposition of an operator. Before the war he had worked in analysis, especially the Navier-Stokes equation, and said that he found in algebraic topology a study that the Nazis would not be able to use in their war effort, in contrast to his expertise as a "mechanic."

It's fair to say that nobody other than Leray understood spectral sequences till well after the war was over. Henri Cartan was a leading figure in post-war mathematical reconstruction. He befriended Leray and helped him explain himself better. He set his students to thinking about Leray's ideas. One was named Jean-Louis Koszul, and it was Koszul who formulated the algebraic object we now call a spectral sequence. Another was Jean-Pierre Serre. Serre wanted to use this method to compute things in homotopy theory proper – homotopy groups, and the cohomology of Eilenberg Mac Lane spaces. He had to recast the theory to work with singular cohomology, on much more general spaces, but in return he considered only what we now call Serre fibrations. This restriction allowed a homotopy-invariant description of the spectral sequence. Leray had used *"anneau spectral"*; Cartan used *"suite de Leray-Koszul"*; and now Serre, in his thesis, brought the two parties together and coined the term *"suite spectral"*. For more history see [22].

> La science ne s'apprend pas: elle se comprend. Elle n'est pas lettre morte et les livres n'assurent pas sa pérennité: elle est une pensée vivante. Pour s'intéresser à elle, puis la

maîtriser, notre eprit doit, habilement guidé, la redécouvrir, de même que notre corps a dû revivre, dans le sien maternel, l'évolution qui créa notre espèce; non point tout ses détails, mais son schéma. Aussi n'y a-t-il qu'une façon efficace de faire acquérir par nos enfants les principes scientifiques qui sont stable, et les procédés techniques qui évoluent rapidement: c'est donner à nos enfants l'esprit de recherche. – Jean Leray [32]

23 The spectral sequence of a filtered complex

We are trying find ways to use a filtration of a space to compute the homology of that space. A simple example is given by the skeleton filtration of a CW complex. Let's recall how that goes. The singular chain complex receives a filtration by sub chain complexes by setting

$$F_sS_*(X) = S_*(\mathrm{Sk}_sX)\,.$$

We then pass to the quotient chain complexes

$$S_*(\mathrm{Sk}_sX, \mathrm{Sk}_{s-1}X) = F_sS_*(X)/F_{s-1}S_*(X)\,.$$

The homology of the sth chain complex in this list vanishes except in dimension s, and the group of cellular s-chains is defined by

$$C_s(X) = H_s(\mathrm{Sk}_sX, \mathrm{Sk}_{s-1}X)\,.$$

In turn, these groups together form a chain complex with differential

$$d : C_s(X) = H_s(\mathrm{Sk}_sX, \mathrm{Sk}_{s-1}X) \xrightarrow{\partial} H_{s-1}(\mathrm{Sk}_{s-1}X) \to H_{s-1}(\mathrm{Sk}_{s-1}X, \mathrm{Sk}_{s-2}X) = C_{s-1}(X)\,.$$

Then $d^2 = 0$ since it factors through two consecutive maps in the long exact sequence of the pair $(\mathrm{Sk}_{s-1}X, \mathrm{Sk}_{s-2}X)$.

We want to think about filtrations

$$\cdots \subseteq F_{s-1}X \subseteq F_sX \subseteq F_{s+1}X \subseteq \cdots X$$

of a space X that don't behave so simply. But the starting point is the same: filter the singular complex accordingly:

$$F_sS_*(X) = S_*(F_sX) \subseteq S_*(X)$$

This is a *filtered (chain) complex.*

To abstract a bit, suppose we are given a chain complex C_* whose homology we wish to compute by means of a filtration

$$\cdots F_{s-1}C_* \subseteq F_sC_* \subseteq F_{s+1}C_* \subseteq \cdots$$

by sub chain complexes. Note that at this point we are allowing the filtration to extend in both directions. And do we need to suppose that the intersection is zero, nor that the union is all of C_*. (And C_* might be nonzero in negative degrees, as well.)

The first step is to form the quotient chain complexes,

$$\mathrm{gr}_sC_* = F_sC_*/F_{s-1}C_*\,.$$

This is a sequence of chain complexes, a *graded* object in the category of chain complexes, and is termed the "associated graded" complex.

What is the relationship between the homologies of these quotient chain complexes and the homology of C_* itself?

We'll set up grading conventions following the example of the filtration by preimages of a skeleton filtration under a fibration, as described in the previous lecture: name the coordinates in the plane (s,t), with the s-axis horizontal and the t-axis vertical. So s will be the filtration degree, and $s+t$ will be the total topological dimension. t is the "complementary degree." This suggests that we should put $\mathrm{gr}_s C_{s+t}$ in bidegree (s,t). Here then is a standard notation:

$$E^0_{s,t} = \mathrm{gr}_s C_{s+t} = F_s C_{s+t}/F_{s-1}C_{s+t}$$

The differential then has bidegree $(0,-1)$. In parallel with the superscript in "E^0," this differential is written d^0.

Next we pass to homology. Let's use the notation

$$E^1_{s,t} = H_{s,t}(E^0_{*,*}, d^0)$$

for the homology of E^0. This in turn supports a differential. In the case of the skeleton filtration, this is the differential in the cellular chain complex. The definition in general is identical:

$$d^1 : E^1_{s,t} = H_{s+t}(F_s/F_{s-1}) \xrightarrow{\partial} H_{s+t-1}(F_{s-1}) \to H_{s+t-1}(F_{s-1}/F_{s-2}) = E^1_{s-1,t}\,.$$

Thus d^1 has bidegree $(-1,0)$. Of course we will write

$$E^2_{s,t} = H_{s,t}(E^1_{*,*}, d^1)\,.$$

In the case of the skeleton filtration, $E^1_{s,t} = 0$ unless $t=0$, and the fact that cellular homology equals singular homology is the assertion that

$$E^2_{s,0} = H_s(X)\,.$$

In general the situation is more complicated because E^1 may be nonzero off the s-axis. So now the magic begins. The claim is that the bigraded group $E^2_{*,*}$ in turn supports a natural differential, written, of course, d^2, this time of bidegree $(-2,1)$; that this pattern continues *ad infinitum*; and that in the end you get (essentially) $H_*(C_*)$. In fact the proof we gave last term that cellular homology agrees with singular homology is no more than a degenerate case of this fact.

Here's the general picture.

Theorem 23.1. *A filtered complex F_*C_* determines a natural* spectral sequence*, consisting of*

- *bigraded abelian groups $E^r_{s,t}$ for $r \geq 0$,*
- *differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ for $r \geq 0$, and*
- *isomorphisms $E^{r+1}_{s,t} \cong H_{s,t}(E^r_{*,*}, d^r)$ for $r \geq 0$,*

such that for $r = 0,1,2$, $(E^r_{*,*}, d^r)$ is as described above, and that under further hypotheses "converges" to $H_*(C_*)$.

Here are further conditions that will suffice to guarantee that the spectral sequence is actually computing $H_*(C_*)$.

Definition 23.2. The filtered complex F_*C_* is *first quadrant* if

- $F_{-1}C_* = 0$,
- $H_n(\mathrm{gr}_s C_*) = 0$ for $n < s$, and
- $C_* = \bigcup F_s C_*$.

Under these conditions, E^1 is zero outside of the first quadrant, and so all the higher "pages" E^r have the same property. It's called a "first quadrant spectral sequence."

The differentials all have total degree -1, but their slopes vary. The longest possibly nonero differential emanating from (s,t) is

$$d^s : E^s_{s,t} \to E^s_{0,t+s-1}\,,$$

and the longest differential attacking (s,t) is

$$d^{t+1} : E^{t+1}_{s+t+1,0} \to E^{t+1}_{s,t}\,.$$

What this says is that for any value of (s,t), the groups $E^r_{s,t}$ stabilize for large r. That stable value is written

$$E^\infty_{s,t}\,.$$

Here's the rest of Theorem 23.1. It uses the natural filtration on $H_*(C_*)$ given by

$$F_s H_n(C_*) = \mathrm{im}(H_n(F_s C_*) \to H_n(C_*))\,.$$

Theorem 23.3. *The spectral sequence of a first quadrant filtered complex* converges to $H_*(C_*)$, *in the sense that*

$$F_{-1}H_*(C_*) = 0\,, \quad \bigcup_s F_s H_*(C_*) = H_*(C_*)\,,$$

and for each s,t there is a natural isomorphism

$$E^\infty_{s,t} \cong \mathrm{gr}_s H_{s+t}(C_*)\,.$$

In symbols, we may write (for any $r \geq 0$)

$$E^r_{*,*} \Longrightarrow H_*(C_*)\,,$$

or, if you want to be explicit about the degrees and which degree is the filtration degree,

$$E^r_{s,t} \underset{s}{\Longrightarrow} H_{s+t}(C_*)\,.$$

Notice right off that this contains the fact that cellular homology computes singular homology: In the spectral sequence associated to the skeleton filtration,

$$\begin{aligned}
E^0_{s,t} =& S_{s+t}(\mathrm{Sk}_s X, \mathrm{Sk}_{s-1} X) \\
E^1_{s,t} =& H_{s+t}(\mathrm{Sk}_s X, \mathrm{Sk}_{s-1} X) = \begin{cases} C_s(X) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \\
E^2_{s,t} =& \begin{cases} H^{\mathrm{cell}}_s(X) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

In a given total degree n there is only one nonzero group left by E^2, namely $E^2_{n,0} = H^{\mathrm{cell}}_n(X)$. Thus no further differentials are possible:

$$E^2_{*,*} = E^\infty_{*,*}\,.$$

The convergence theorem then implies that

$$\mathrm{gr}_s H_n(X) = \begin{cases} E^\infty_{n,0} = H^{\mathrm{cell}}_n(X) & \text{if } s = n \\ 0 & \text{otherwise} \end{cases}$$

So the filtration of $H_n(X)$ changes only once:

$$0 = \cdots = F_{n-1}H_n(X) \subseteq F_n H_n(X) = \cdots = H_n(X)\,,$$

and

$$F_n H_n(X)/F_{n-1}H_n(X) = E^\infty_{n,0} = H^{\mathrm{cell}}_n(X)\,.$$

So

$$H_n(X) = H^{\mathrm{cell}}_n(X)\,.$$

Before we explain how to construct the spectral sequence, let me point out one corollary at the present level of generality.

Corollary 23.4. *Let $f : C \to D$ be a map of first quadrant filtered chain complexes. If $E^r_{*,*}(f)$ is an isomorphism for some r, then $f_* : H_*(C) \to H_*(D)$ is an isomorphism.*

Proof. The map $E^r(f)$ is an isomorphism which is also also a chain map, i.e., it is compatible with the differential d^r. It follows that $E^{r+1}(f)$ is an isomorphism. By induction, we conclude that $E^\infty_{s,t}(f)$ is an isomorphism for all s, t. By Theorem 23.3, the map $\mathrm{gr}_s(f_*) : \mathrm{gr}_s H_*(C) \to \mathrm{gr}_s H(D)$ is an isomorphism. Now the conditions in Definition 23.2) let us use induction and the five lemma to conclude the proof. □

Direct construction

In a later lecture I will describe a structure known as an "exact couple" that provides a construction of a spectral sequence that is both clean and flexible. But the direct construction from a filtered complex has its virtues as well. Here it is. The detailed computations are annoying but straightforward.

Define the following subspaces of $E^0_{s,t} = F_s C_{s+t}/F_{s-1}C_{s+t}$, for $r \geq 1$.

$$\begin{aligned} Z^r_{s,t} =& \{c : \exists\, x \in c \text{ such that } dx \in F_{s-r}C_{s+t-1}\}\,, \\ B^r_{s,t} =& \{c : \exists\, y \in F_{s+r-1}C_{s+t+1} \text{ such that } dy \in c\}\,. \end{aligned}$$

So an "r-cycle" is a class that admits a representative whose boundary is r filtrations smaller; the larger r is the closer the class is to containing an actual cycle. An "r-boundary" is a class admitting a representative that is a boundary of an element allowed to lie in filtration degree $r-1$ stages larger. When $r = 1$, these are exactly the cycles and boundaries with respect to the differential d^0 on $E^0_{*,*}$.

We have inclusions

$$B^1_{*,*} \subseteq B^2_{*,*} \subseteq \cdots \subseteq Z^2_{*.*} \subseteq Z^1_{*,*}$$

and define

$$E^r_{s,t} = Z^r_{s,t}/B^r_{s,t}\,.$$

These pages are successively smaller groups of cycles modulo successively larger subgroups of boundaries. The differential d^r is of course induced from the differential d in C_*, and $H_{s,t}(E^r_{*,*}, d^r) \cong E^{r+1}_{s,t}$. In the first quadrant situation, the r-boundaries and the r-cycles stabilize to

$$\begin{aligned} Z^\infty_{s,t} &= \{c : \exists\, x \in c \text{ such that } dx = 0\}\,, \\ B^\infty_{s,t} &= \{c : \exists\, y \in C_{s+t+1} \text{ such that } dy \in c\}\,. \end{aligned}$$

The quotient, $E^\infty_{s,t}$, is exactly $F_s H_{s+t}(C_{*,*})/F_{s-1}H_{s+t}(C_{*,*})$.

24 Serre spectral sequence

Fix a fibration $p : E \to B$, with B a CW-complex. We obtain a filtration on E by taking the preimage of the s-skeleton of B: $E_s = p^{-1}\mathrm{Sk}_s B$. This induces a filtration on $S_*(E)$ given by

$$F_s S_*(E) = S_*(p^{-1}\mathrm{Sk}_s(B)) \subseteq S_*(E)\,.$$

The spectral sequence resulting from Theorem 23.1 is the *Serre spectral sequence.*

This was not Serre's construction [35], by the way; he did not employ a CW structure at all, but rather worked directly with a singular theory – but rather than simplices, he used cubes, which are well adapted to the study of bundles since a product of cubes is again a cube. We will describe a variant of Serre's construction in a later lecture, one that is technically easier to work with and that makes manifest important multiplicative features of the spectral sequence. We will not try to dot all the i's in the construction we describe in this lecture, and for simplicity we'll imagine that p is actually a fiber bundle.

In this spectral sequence,

$$E^1_{s,t} = H_{s+t}(F_s E, F_{s-1}E)\,.$$

Pick a cell structure

$$\begin{array}{ccc} \coprod_{i\in\Sigma_s} S^{s-1}_i & \longrightarrow & \coprod_{i\in\Sigma_s} D^s_i \\ \downarrow & & \downarrow \\ B_{s-1} & \longrightarrow & B_s \end{array}$$

Let $\alpha : D^s_i \to B_s$ be characteristic map, and let F_i be the fiber over the center of e^s_i in B. The pullback of $E \downarrow B$ under α is a trivial fibration since D^s_i is contractible. Now

$$\coprod_{i\in\Sigma_s} (D^s_i, S^{s-1}_i) \times F_i \to (F_s E, F_{s-1}E)$$

is a relative homeomorphism, so by excision

$$E^1_{s,t} = H_{s+t}(F_s E, F_{s-1}E) = \bigoplus_{i\in\Sigma_s} H_{s+t}((D^s_i, S^{s-1}_i) \times F_i) = \bigoplus_{i\in\Sigma_s} H_t(F_i)\,.$$

In particular, this filtration satisfies the requirements of Definition 23.2, since $H_t(F_i) = 0$ for $t < 0$. We have a convergent spectral sequence. It remains to work out what d^1 is. I won't do this in detail but I'll tell you how it turns out.

It's important to appreciate that the fibers F_i vary from one cell to the next. If B is not path-connected, these fibers don't even have to be of the same homotopy type. If B *is* path connected, then they do, but the homotopy equivalence is determined by a homotopy class of paths from one center to the other and so is not canonical. If B is not simply connected, the functor

$$p^{-1}(-) : \Pi_1(B) \to \mathrm{Ho}(\mathbf{Top})$$

may not be constant. But at least we see that the fibration defines functors

$$H_t(p^{-1}(-)) : \Pi_1(B) \to \mathbf{Ab} \quad \text{with} \quad b \mapsto H_t(p^{-1}(b))\,.$$

This is, or determines, a *local coefficient system*. We encountered these before, in our exploration of orientability. There a "local coefficient system" was a covering space with continuously varying abelian group structures on the fibers. If the space is path connected and semi-locally simply connected, there is a universal cover, and giving a covering space is equivalent to giving an action of the fundamental group on a set. We can free this equivalence from dependence on path connectedness (and choice of basepoint) by speaking of functors from the fundamental groupoid to abelian groups. CW complexes are locally contractible [12, e.g. Appendix on CW complexes, Proposition 4] and so this equivalence applies in our case.

If this local system is in fact constant (for example if B is simply connected) the differential in E^1 is none other than the cellular differential in

$$C_*(B; H_t(F))$$

(where we write F for any fiber), and so

$$E^2_{s,t} = H_s(B; H_t(F))\,.$$

This is the case we will mostly be concerned with. But the general case is the same, with the understanding that we mean homology of B with coefficients in the local system $H_t(p^{-1}(-))$.

Here's a base-point dependent way of thinking of how to compute homology or cohomology of a space with coefficients in a local system. We assume that our space X is path-connected and nice enough to admit a universal cover $\widetilde{X}$. Pick a basepoint $*$. Giving a local coefficient system is the same as giving a $\mathbb{Z}[\pi_1(X,*)]$-module. Write M for both. The fundamental group acts on $\widetilde{X}$ and so on its singular chain complex. Now we can say that

$$H_*(X;M) = H_*(S_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,*)]} M)\,, \quad H^*(X;M) = H^*(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,*)]}(S_*(\widetilde{X}), M)\,.$$

Here's the general result.

Theorem 24.1. *Let $p : E \to B$ be a Serre fibration, R a commutative ring, and M an R-module. There is a first quadrant spectral sequence of R-modules with*

$$E^2_{s,t} = H_s(B; H_t(p^{-1}(-); M))$$

that converges to $H_*(E;M)$. It is natural from E^2 on for maps of fibrations.

This theorem expresses one important perspective on spectral sequences: They can serve to implement a "local-to-global" strategy. A fiber bundle is locally a product. The spectral sequence explains how the "local" (in the base) homology of E gets integrated to produce the "global" homology of E itself.

Loops on spheres

Here's a first application of the Serre spectral sequence: a computation of the homology of the space of pointed loops on a sphere, ΩS^n. It is the fiber of the fibration $PS^n \to S^n$, where PS^n is the space of pointed maps $(S^n)^I_*$. The space PS^n is contractible, by the spaghetti move.

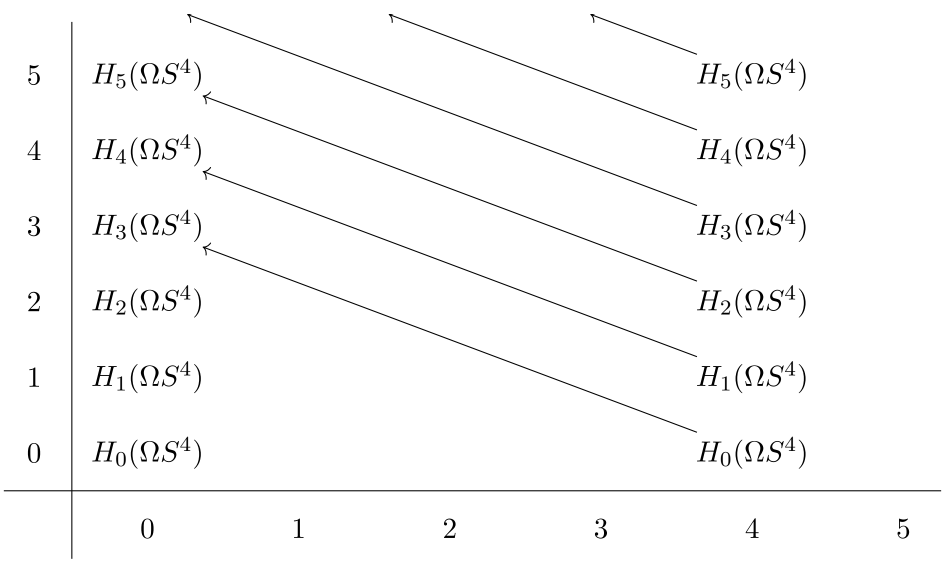
It is often said that the Serre spectral sequence is designed to compute the homology of the total space starting with the homologies of the fiber and of the base. This is not true! Rather, it establishes a relationship between these three homologies, one that can be used in many different ways. Here we know the homology of the total space (since PS^n is contractible) and of the base, and we want to know the homology of the fiber.

The case $n = 1$ is special: S^1 is a Eilenberg Mac Lane space $K(\mathbb{Z}, 1)$, so ΩS^1 is weakly equivalent to the discrete space $\mathbb{Z}$.

So suppose $n \geq 2$. Then the base is simply connected and torsion-free, so in the Serre spectral sequence

$$E^2_{s,t} = H_s(S^n; H_t(\Omega S^n)) = H_s(S^n) \otimes H_t(\Omega S^n)\,.$$

Here's a picture, for $n = 4$.



As you can see, the only possible nonzero differentials are of the form

$$d^n : E^n_{n,t} \to E^n_{0,t+n-1}\,.$$

So $E^2_{*,*} = E^{n-1}_{*,*}$ and $E^{n+1}_{*,*} = E^\infty_{*,*}$.

The spectral sequence converges to $H_*(PS^n)$, which is $\mathbb{Z}$ in dimension 0 and 0 elsewhere. This immediately implies that

$$H_t(\Omega S^n) = 0 \quad \text{for} \quad 0 < t < n-1$$

since nothing could kill these groups on the fiber.

The fiber is path connected, $H_0(\Omega S^n) = \mathbb{Z}$, so we know the bottom row in E^2. $E^2_{n,0}$ must die. It can't be killed by being hit by a differential, since everything below the s-axis is trivial (and also because everything to its right is trivial). So it must die by virtue of d^n being injective on it. In fact that differential must be an isomorphism, since if it fails to surject onto $E^n_{0,n-1}$ there would be something left in $E^{n+1}_{0,n-1} = E^\infty_{0,n-1}$, and it would contribute nontrivially to $H_{n-1}(PS^n) = 0$.

This language of mortal combat gives extra meaning to the "spectral" in "spectral sequence."

So $H_{n-1}(\Omega S^n) = \mathbb{Z}$. This feeds back into the spectral sequence: $E^2_{n,n-1} = \mathbb{Z}$. Now that class has to kill or be killed. It can't be killed because everything to its right is zero, so d^n must be injective on it. And it must surject onto $E^n_{0,2(n-1)}$, for the same reason as before.

This establishes the inductive step. We have shown that all the d^n's are isomorphisms (except the ones involving $E^n_{0,0}$), and established:

Proposition 24.2. *Let $n \geq 2$. Then*

$$H_t(\Omega S^n) = \begin{cases} \mathbb{Z} & \text{if} \quad (n-1)|t \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

Evenness

Sometimes it's easy to see that a spectral sequence collapses. For example, suppose that

$$E^r_{s,t} = 0 \text{ unless both } s \text{ and } t \text{ are even}.$$

Then all differentials in E^r and beyond must vanish, because they all have total degree -1. Actually all that is needed for this argument is that $E^r_{s,t} = 0$ unless $s+t$ is even. There may still be extension problems, though.

25 Exact couples

Today I would like to show you a very simple piece of linear algebra called an *exact couple*. A filtered complex gives rise to an exact couple, and an exact couple gives rise to a spectral sequence. Exact couples were discovered by Bill Massey (1920–2017, Professor at Yale) independently of the French development of spectral sequences.

Definition 25.1. An *exact couple* is a diagram of abelian groups

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

that is exact at each node.

As $jkjk = 0$, the map $jk : E \to E$ is a differential, denoted d.

An exact couple determines a "derived couple"

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

where

$$A' = \mathrm{im}(i) \quad \text{and} \quad E' = H(E, d).$$

Iterating this procedure, we get a sequence of exact couples

$$\begin{tikzcd} A^{(r)} \arrow[rr, "i^{(r)}"] & & A^{(r)} \arrow[dl, "j^{(r)}"] \\ & E^{(r)} \arrow[ul, "k^{(r)}"] & \end{tikzcd}$$

If we impose appropriate gradings, the "E" terms will form a spectral sequence.

We have to explain the maps in the derived couple.

i': this is just i restricted to $A' = \mathrm{im}(i)$. Obviously i carries $\mathrm{im}(i)$ into $\mathrm{im}(i)$.

j': Note that ja is a cycle in E: $dja = jkja = 0$. Define

$$j'(ia) = [ja] \,.$$

To see that this is well defined, we need to see that if $ia = 0$ then ja is a boundary. By exactness there is an element $e \in E$ such that $ke = a$. Then $de = jke = ja$.

k': Let $e \in E$ be a cycle. Since $0 = de = jke$, $ke \in \mathrm{im}(i) = A'$ by exactness. Define

$$k'([e]) = ke \,.$$

To see that this is well defined, suppose that $e = de'$. Then $ke = kde' = kjke' = 0$.

Exercise 25.2. Check that these maps indeed yield an exact couple.

Gradings

Now suppose we are given a filtered complex. It will define an exact couple in which A is given by the homology groups of the filtration degrees and E is given by the homology groups of the associated quotient chain complexes.

In order to accommodate this example we need to add gradings – in fact, bigradings. Here's the relevant definition.

Definition 25.3. An exact couple of bigraded abelian groups is *of type* r if the structure maps have the following bidegrees.

$$\begin{aligned} ||i|| =& (1,-1) \\ ||j|| =& (0,0) \\ ||k|| =& (-r, r-1) \end{aligned}$$

It's clear from this that $||d|| = ||jk|| = (-r, r-1)$, the bidegree appropriate for the rth stage of a spectral sequence. We should specify the gradings on the abelian groups in the derived couple. Define $A'_{s,t}$ to sit in the factorization

$$\begin{tikzcd} A_{s,t} \arrow[rr, "i"] \arrow[dr, two heads] & & A_{s+1,t-1} \\ & A'_{s,t} \arrow[ur, hook] & \end{tikzcd}$$

and $E'_{s,t} = H_{s,t}(E_{*,*})$. Then if $e \in E_{s,t}$, $ke \in A_{s-r,t+r-1}$, but if e is a cycle then ke lies in the subgroup $A'_{s-r-1,t-r}$, so $||k'|| = (r+1, -r)$: the derived couple is of type $(r+1)$.

Given a filtered complex

$$\cdots \subseteq F_{s-1}C_* \subseteq F_sC_* \subseteq F_{s+1}C_* \subseteq \cdots \,,$$

define

$$A^1_{s,t} = H_{s+t}(F_sC_*) \quad , \quad E^1_{s,t} = H_{s+t}(\mathrm{gr}_sC_*) \,.$$

This agrees with our earlier use of the notation $E^1_{s,t}$. The structure maps are given in the obvious way: i^1 is induced by the inclusion of one filtration degree into the next (and has bidegree $(1,-1)$); j^1 is induced from the quotient map (and has bidegree $(0,0)$); and k^1 is the boundary homomorphism in the homology long exact sequence (and has bidegree $(-1,0)$).

Given any exact couple of type 1, (A^1, E^1), we'll write

$$A^r = (A^1)^{(r-1)}\,, \quad E^r = (E^1)^{(r-1)}$$

for the $(r-1)$ times derived exact couple, which is of type r.

Differentials

An exact couple can be unfolded in a series of linked exact triangles, like this (taking $r = 1$ for concreteness, and omitting the second index):

$$\begin{tikzcd}
\cdots \arrow[rr, "i"] & & A^1_{s-3} \arrow[dl, "j"] \arrow[rr, "i"] & & A^1_{s-2} \arrow[dl, "j"] \arrow[rr, "i"] & & A^1_{s-1} \arrow[dl, "j"] \arrow[rr, "i"] & & A^1_s \arrow[dl, "j"] \arrow[r, "i"] & \cdots \\
\cdots & E^1_{s-3} \arrow[ul, "k"] \arrow[ur, phantom, "\circ"] & & E^1_{s-2} \arrow[ll, "d^1"] \arrow[ul, "k"] \arrow[ur, phantom, "\circ"] & & E^1_{s-1} \arrow[ll, "d^1"] \arrow[ul, "k"] \arrow[ur, phantom, "\circ"] & & E^1_s \arrow[ll, "d^1"] \arrow[ul, "k"] \arrow[ur, phantom, "\circ"] & & \cdots
\end{tikzcd}$$

The triangles marked with $\circ$ are exact; the lower ones commute, and define d^1.

This image is useful in understanding the differentials in the associated spectral sequence. Start with an element $x \in E^1_s$. Suppose it's a cycle. Then its image $kx \in A^1_{s-1}$ is killed by j and hence pulls back under i, to, say, $x_1 \in A^1_{s-2}$. The image in E^1_{s-2} of x_1 under j is a representative for $d^2[x]$. Suppose that $d^2[x] = 0$. Then we can improve the lift x_1 to one that pulls back one step further, to, say, $x_2 \in A^1_{s-3}$; and $d^3[x] = [jx_2]$. This pattern continues. The further you can pull kx back, the longer x survives in the spectral sequence. If it pulls back forever, then you appeal to a convergence condition to conclude that $kx = 0$, and x therefore lifts under j to an element $\overline{x}$ in A^1_s. The direct limit

$$L = \varinjlim(\cdots \to A^1_s \to A^1_{s+1} \to A^1_{s+2} \to \cdots)$$

is generally what one is interested in (it's $H_*(C_*)$ in the first quadrant filtered complex situation, for example) and one may say that "x survives to" the image of $\overline{x}$ in L.

Other examples

Topology is inhabited by many spectral sequences that do not arise from a filtered complex. For example, if you have a tower of fibrations, you get an exact couple by linking together the homotopy long exact sequences of the individual fibrations. Well, almost. The problem is what happens at the bottom: groups may not be abelian, or even groups; and even if they are, you may not be able to guarantee exactness at π_0. For example, form the Whitehead tower of a space Y and map some well-pointed space X into it. We get a new tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & (\tau_{\geq 2}Y)^X_* \arrow[r] \arrow[d] & K(\pi_2(Y),2)^X_* \\
 & (\tau_{\geq 1}Y)^X_* \arrow[r] \arrow[d] & K(\pi_1(Y),1)^X_* \\
Y^X_* \arrow[r, phantom, "="] & (\tau_{\geq 0})^X_* \arrow[r] & K(\pi_0(Y),0)^X_* \,.
\end{tikzcd}$$

The homotopy groups of the spaces on the right form the E^1-term, and are easy to compute:

$$\pi_n(K(\pi_p(Y),p)^X_*) = [S^n \wedge X, K(\pi_p(Y),p)]_* = [X, K(\pi_p(Y), p-n)]_* = \overline{H}^{p-n}(X;\pi_p(Y))\,.$$

Insofar as this is a spectral sequence at all, the E^1 term is given by

$$E^1_{s,t} = \overline{H}^{-2s-t}(X;\pi_{-s}(Y,*))\,.$$

It's concentrated between the lines $t=-s$ and $t=-2s$, in the second quadrant of the plane.

This picture is very closely related to obstruction theory, and indeed obstruction theory can be set up using it. Its failings as a spectral sequence can be repaired in various ways I won't discuss. If it can be repaired, the spectral sequence converges to $\pi_*(Y^X_*)$, or wants to.

For another example, there are many "generalized homology theories" – sequences of functors satisfying the Eilenberg-Steenrod axioms other than the dimension axiom – K-theory, bordism theories, and many others. Write $R_*(-)$ for any such theory. The skeleton filtration construction of the Serre spectral sequence can be applied to compute the R-homology of the total space of a fibration $p: E \to B$: To construct the exact couple, all you need is the long exact sequence of a pair, which is available in R-homology. You find for each t a local coefficient system $R_t(p^{-1}(-))$, and

$$E^2_{s,t} = H_s(B; R_t(p^{-1}(-))) \underset{s}{\Longrightarrow} R_{s+t}(E)$$

Even the case $p : E \xrightarrow{=} B$ is interesting: then the local coefficient system is guaranteed to be trivial, and we get

$$E^2_{s,t} = H_s(E; R_t(*)) \underset{s}{\Longrightarrow} R_{s+t}(E)\,.$$

This is the "Atiyah-Hirzebruch spectral sequence," and it provides a powerful tool for computing these generalized homology theories.

Both of these spectral sequences require us to move out of the first quadrant setting. The Atiyah-Hirzebruch-Serre spectral sequence can fill up the right half-plane.

26 The Gysin sequence, edge homomorphisms, and the transgression

Now we'll discuss a general situation, a common one, that displays many of the ways in which the Serre spectral sequence relates the homology groups of fiber, total space, and base.

Suppose $p : E \to B$ is a fibration; assume the base is path-connected, and that the fiber has homology isomorphic to that of S^{n-1} with $n > 1$. Let us use the Serre spectral sequence to determine how the homologies of E and of B are related. We will assume that this "spherical fibration" is orientable, and choose an orientation. This means that the local coefficient system $H_{n-1}(p^{-1}(-))$ is trivial, and provided with a trivialization: a preferred generator of $H_{n-1}(p^{-1}(b))$ that varies continuously with $b \in B$. For example, we might be looking at $S^{2k-1} \downarrow \mathbb{C}P^{k-1}$ or $S^{4k-1} \downarrow \mathbb{H}P^{k-1}$, or the complement of the zero-section in the tangent bundle of an oriented n-manifold.

There are just two nonzero rows in this spectral sequence. This means that there's just one possibly nonzero differential:

$$E^2_{*,*} = E^3_{*,*} = \cdots = E^n_{*,*}\,;$$

then a differential

$$d^n : E^n_{s,0} \to E^n_{s-n,n-1}$$

occurs; and then

$$E^{n+1}_{*,*} = \cdots = E^\infty_{*,*}\,.$$

Taking homology with respect to d^n gives the top row of

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{s,0} \arrow[r] \arrow[dr] & E^n_{s,0} \arrow[r, "d^n"] \arrow[d, "="] & E^n_{s-n,n-1} \arrow[r] \arrow[d, "="] & E^\infty_{s-n,n-1} \arrow[r] & 0 \\
 & & H_s(B) \arrow[r] & H_{s-n}(B) \arrow[ur] & &
\end{tikzcd}$$

To explain the rest of this diagram, path connectedness of S^{n-1} gives the isomorphism

$$E^n_{s,0} = E^2_{s,0} = H_s(B)\,,$$

and the oriention determines

$$E^n_{s-n,n-1} = E^2_{s-n,n-1} = H_{s-n}(B; H_{n-1}(S^{n-1})) = H_{s-n}(B)\,.$$

Now look at total degree n. The filtration of $H_n(E)$ changes at most twice, with associated quotients given by the E^∞ term: so there is a short exact sequence

$$0 \to E^\infty_{s-n+1,n-1} \to H_s(E) \to E^\infty_{s,0} \to 0\,.$$

These two families of exact sequences splice together to give a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[dr] & & & 0 \arrow[d] & & & \\
\cdots \arrow[r] & H_{s+1}(B) \arrow[r, "d^n"] & H_{s-n+1}(B) \arrow[r] \arrow[dr] & E^\infty_{s-n+1,n-1} \arrow[r] \arrow[d] & 0 & & \\
 & & & H_s(E) \arrow[d] \arrow[dr] & & & \\
 & & 0 \arrow[r] & E^\infty_{s,0} \arrow[r] \arrow[d] & H_s(B) \arrow[r, "d^n"] & H_{s-n}(B) \arrow[r] \arrow[dr] & \cdots \\
 & & & 0 & & & \cdots
\end{tikzcd}$$

Proposition 26.1. *Let $p : E \to B$ be a Serre fibration whose fiber is a homology $(n-1)$-sphere, and assume it is oriented (so the local coefficient system $H_{n-1}(p^{-1}(-))$ is trivialized). There is a naturally associated long exact sequence, the* Gysin sequence

$$\cdots \to H_{s+1}(B) \to H_{s-n+1}(B) \to H_s(E) \xrightarrow{p_*} H_s(B) \to H_{s-n}(B) \to \cdots .$$

(Werner Gysin (1915-1998) described this in his thesis at ETH under Heinz Hopf.) The only part of this that we have not proven is that the middle map here is in fact the map induced by the projection p. That's the story of "edge homomorphisms," which we take up next.

First, though, and example. The Gysin sequence of the S^1-bundle $S^\infty \downarrow \mathbb{C}P^\infty$ looks like this:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
0 \arrow[r] & H_4(\mathbb{C}P^\infty) \arrow[r] & H_2(\mathbb{C}P^\infty) \arrow[dll] \\
0 \arrow[r] & H_3(\mathbb{C}P^\infty) \arrow[r] & H_1(\mathbb{C}P^\infty) \arrow[dll] \\
0 \arrow[r] & H_2(\mathbb{C}P^\infty) \arrow[r] & H_0(\mathbb{C}P^\infty) \arrow[dll] \\
0 \arrow[r] & H_1(\mathbb{C}P^\infty) \arrow[r] & 0 \arrow[dll] \\
H_0(S^\infty) \arrow[r] & H_0(\mathbb{C}P^\infty) \arrow[r] & 0 .
\end{tikzcd}$$

Working inductively up the tower, you compute what we know:

$$H_n(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & \text{if } 2|n \geq 0 \\ 0 & \text{otherwise} . \end{cases}$$

Edge homomorphisms

In the Serre spectral sequence for the fibration $p : E \to B$, what can we say about the evolution of the bottom edge, or of the left edge? Let's assume that the fiber is path connected and that the local coefficient system is trivial, so in

$$E^2_{s,t} = H_s(B; H_t(F)) \underset{s}{\Longrightarrow} H_{s+t}(E)$$

the bottom edge is canonically isomorphic to $H_*(B)$.

Being at the bottom, no nontrivial differentials can ever hit it. So the successive process of taking homology will be a succession of taking kernels:

$$E^{r+1}_{n,0} = \ker(d^r : E^r_{n,0} \to E^r_{n-r,r-1}) .$$

Of course when $r > s$ things quiet down. So

$$E^2_{n,0} \supseteq E^3_{n,0} \supseteq \cdots \supseteq E^{n+1}_{n,0} = E^\infty_{n,0} .$$

Now $H_n(E)$ enters the picture, along with its filtration. The whole of $H_n(E)$ is already hit by $H_n(p^{-1}\mathrm{Sk}_n B)$. This is confirmed by the fact that the associated graded $\mathrm{gr}_s H_n(E) = E^\infty_{s,n-s}$ vanishes for $s > n$. So $F_n H_n(E) = H_n(E)$.

Putting all this together, we get a map

$$H_n(E) = F_n H_n(E) \twoheadrightarrow \mathrm{gr}_n H_n(E) = E^\infty_{n,0} = E^{n+1}_{n,0} \hookrightarrow E^n_{n,0} \hookrightarrow \cdots \hookrightarrow E^2_{n,0} = H_n(B)\,.$$

This composite is an *edge homomorphism* for the spectral sequence. It's something you can define for any first quadrant filtered complex. In the Serre spectral sequence case, it has a direct interpretation:

Proposition 26.2. *This edge homomorphism coincides with the map $p_* : H_n(E) \to H_n(B)$.*

This explains the role of the differentials off the bottom row of the spectral sequence. They are obstructions to classes lifting to the homology of the total space. This reflects the intuition we tried to develop several lectures ago. The image of $p_* : H_n(E) \to H_n(B)$ is precisely the intersection (so to speak) of the kernels of the differentials coming off of $E^2_{n,0}$.

Before we prove this, let's notice that there is a dual picture for the vertical axis. Now all differentials leaving $E^r_{0,n}$ are trivial, so we get surjections

$$E^2_{0,n} \twoheadrightarrow E^3_{0,n} \twoheadrightarrow \cdots \twoheadrightarrow E^{n+2}_{0,n} = E^\infty_{0,n}\,.$$

On the other hand, the smallest nonzero filtration degree of $H_n(E)$ is $F_0 H_n(E)$. Thus we have another "edge homomorphism,"

$$H_n(F) = E^2_{0,n} \twoheadrightarrow E^3_{0,n} \twoheadrightarrow \cdots \twoheadrightarrow E^{n+2}_{0,n} = E^\infty_{0,n} = F_0 H_n(E) \hookrightarrow H_n(E)\,.$$

Proposition 26.3. *This edge homomorphism coincides with the map $i_* : H_n(F) \to H_n(E)$ induced by the inclusion of the fiber.*

So the kernel of i_* is union of the images (so to speak) of the differentials coming into $E^2_{0,n}$. These represent chains in E which serve as null-homologies of cycles in F.

Proof of Propositions 26.2 and 26.3. The map of fibrations

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & * \arrow[d] \\
E \arrow[r] \arrow[d] & B \arrow[d] \\
B \arrow[r] & B
\end{tikzcd}$$

induces a commutative diagram in which the top and bottom arrows are edge homomorphisms:

$$\begin{tikzcd}
H_n(E) \arrow[r] \arrow[d, "p_*"] & H_n(B) \arrow[d, "(1_B)_*"] \\
H_n(B) \arrow[r] & H_n(B)\,.
\end{tikzcd}$$

So we just need to check that the bottom edge homomorphism associated to the identity fibration $1_B : B \to B$ is the identity map $H_n(B) \to H_n(B)$. This I leave to you.

The proof of Proposition 26.3 is similar. $\square$

Very often you begin with some homomorphism, and you are interested in whether it is an isomorphism, or how it can be repaired to become an isomorphism. If you can write it as an edge homomorphism in a spectral sequence, then you can regard the spectral sequence as measuring how far from being an isomorphism your map is; it provides the reasons why the map fails to be either injective or surjective.

Transgression

There is a third aspect of the Serre spectral sequence that deserves attention, namely, the differential going clear across the spectral sequence, all the way from base to fiber. We'll study it in case the fiber and the base are both path connected and the local coefficient systems $H_t(p^{-1}(-))$ are trivial. Write F for the fiber.

The differentials

$$d^n : E^n_{n,0} \to E^n_{0,n-1}$$

are known as *transgressions*, and an element of $E^2_{n,0} = H_n(B)$ that survives to $E^n_{n,0}$ is said to be *transgressive*. The first one is a homomorphism

$$d^2 : H_2(B) \to H_1(F)\,,$$

but after that d^n is merely an *additive relation* between $H_n(B)$ and $H_{n-1}(F)$: It has a *domain of definition*

$$E^s_{n,0} \subseteq E^2_{n,0} = H_n(B)$$

and *indeterminacy*

$$\ker(H_{n-1}(F) = E^2_{0,n-1} \twoheadrightarrow E^n_{0,n-1})\,.$$

Let me expand on what I mean by an additive relation. A good reference is [19, II §6].

Definition 26.4. An *additive relation* $R : A \rightharpoonup B$ is a subgroup $R \subseteq A \times B$.

For example the graph of a homomorphism $A \to B$ is an additive relation. Additive relations compose in the evident way: the composite of $R : A \rightharpoonup B$ with $S : B \rightharpoonup C$ is

$$\{(a,c) : \exists\, b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\} \subseteq A \times C\,.$$

Every additive relation has a "converse,"

$$R^{-1} = \{(b,a) : (a,b) \in R\} : B \rightharpoonup A\,.$$

An additive relation has a *domain*

$$D = \{a \in A : \exists\, b \in B \text{ such that } (a,b) \in R\} \subseteq A$$

and an *indeterminancy*

$$I = \{b \in B : (0,b) \in R\}\,,$$

and determines a homomorphism

$$f : D \to B/I$$

by

$$f(a) = b + I \text{ for } b \in B \text{ such that } (a,b) \in R\,.$$

Conversely, such a triple (D, I, f) determines an additive relation,

$$R = \{(a, b) : a \in D \text{ and } b \in f(a)\}\,.$$

An additive relation is defined as a subspace of $A \times B$, but any "span"

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

determines one by taking the image of the resulting map $C \to A \times B$.

End of digression. We have the transgression $d^n : H_n(B) \rightharpoonup H_{n-1}(F)$. Another such additive relation is determined by the span

$$H_n(B) = H_n(B, *) \xleftarrow{p_*} H_n(E, F) \xrightarrow{\partial} H_{n-1}(F)\,.$$

Proposition 26.5. *These two linear relations coincide.*

Proof sketch. This phenomenon is actually how we began our discussion of spectral sequences. Let $x \in H_n(B)$. Since $n > 0$ we can just as well regard it as a class in $H_n(B, *)$. Represent it by a cycle $c \in Z_n(B, *)$. (In the Hopf fibration case this simplifies the representative by making the constant cycle optional.) Lift it to a chain in the total space E. In general, this chain will not be a cycle (consider the Hopf fibration). The differentials record this boundary; let us recall the explicit construction of the differential in §**??**. Saying that the class x survives to E^n is the same as saying that we can find a lift to a chain c in E, with $dc \in S_{n-1}(F)$, that is, to a relative cycle in $S_{n-1}(E, F)$. Then $d^n(x)$ is represented by the class $[dc] \in H_{n-1}(F)$. This is precisely the transgression. □

27 The Serre exact sequence and the Hurewicz theorem

Serre exact sequence

Suppose $\pi : E \to B$ is a fibration over a path-connected base. Pick a point $* \in E$, use its image $* \in B$ as a basepoint in B, write $F = \pi^{-1}(*) \subseteq E$ for the fiber over $*$, and equip it with the point $* \in E$ as a basepoint. Suppose also that F is path connected.

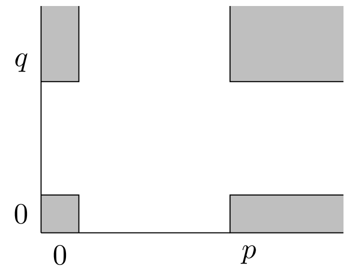
Pick a coefficient ring R. Everything we've done works perfectly with coefficients in R – all abelian groups in sight come equipped with R-module structures. Let's continue to suppress the coefficient ring from the notation. Suppose that the low-dimensional homology of both fiber and base vanishes:

$$\begin{aligned} H_s(B) &= 0 \quad \text{for} \quad 0 < s < p \\ H_t(F) &= 0 \quad \text{for} \quad 0 < t < q\,. \end{aligned}$$

Assume that $\pi_1(B, *)$ act trivially on $H_*(F)$, so the Serre spectral sequence (now with coefficients in R!) takes the form

$$E^2_{s,t} = H_s(B; H_t(F)) \underset{s}{\Longrightarrow} H_{s+t}(E)\,.$$

Our assumptions imply that $E^2_{0,0} = R$ is all alone; otherwise everything with $s < p$ vanishes and everything with $t < q$ vanishes.



For a while, the only possibly nonzero differentials are the transgressions

$$d^s : E^s_{s,0} \to E^s_{0,s-1} \,.$$

The result, in this range, is an exact sequence

$$0 \to E^\infty_{s,0} \to H_s(B) \xrightarrow{d^s} H_{s-1}(F) \to E^\infty_{0,s-1} \to 0 \,.$$

Again, in this range, these end terms are the only two possibly nonzero associated quotients in $H_n(E)$ – there is a short exact sequence

$$0 \to E^\infty_{0,n} \to H_n(E) \to E^\infty_{n,0} \to 0 \,.$$

– and splicing things together we arrive again at a long exact sequence

$$\begin{tikzcd}
H_{p+q-1}(F) \arrow[r, "i_*"] & H_{p+q-1}(E) \arrow[r, "p_*"] & H_{p+q-1}(B) \arrow[dll] \\
H_{p+q-2}(F) \arrow[r, "i_*"] & H_{p+q-2}(E) \arrow[r, "p_*"] & H_{p+q-2}(B) \arrow[dll] \\
H_{p+q-3}(F) \arrow[r, "i_*"] & \cdots \,.
\end{tikzcd}$$

This is the *Serre exact sequence*: in this range of dimensions homology and homotopy behave the same! We can't extend it further to the left because the kernel of the edge homomorphism $H_{p+q-1}(F) \to H_{p+q-1}(E)$ has two sources: the image of $d^p : E^p_{p,q} \to E^p_{0,p+q-1}$, and the image of $d^{p+q} : E^{p+q}_{p+q,0} \to E^{p+q}_{0,p+q-1}$.

Comparison with homotopy

The Serre exact sequence mimics the homotopy long exact sequence of the fibration.

Proposition 27.1. *The Hurewicz map participates in a commutative ladder*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{p+q-1}(F) \arrow[r, "i_*"] \arrow[d, "h"] & \pi_{p+q-1}(E) \arrow[r, "\pi_*"] \arrow[d, "h"] & \pi_{p+q-1}(B) \arrow[r] \arrow[d, "h"] & \pi_{p+q-2}(F) \arrow[r] \arrow[d, "h"] & \cdots \\
 & H_{p+q-1}(F) \arrow[r, "i_*"] & H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[r] & H_{p+q-2}(F) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The left two squares commutes by naturality of the Hurewicz map. The right square commutes because, according to our geometric interpretation of the transgression, both boundary maps arise in the same way:

$$\begin{tikzcd}
\pi_n(B) \arrow[d, "h"] & \pi_n(E,F) \arrow[l, "\cong"'] \arrow[r, "\partial"] \arrow[d, "h"] & \pi_{n-1}(F) \arrow[d, "h"] \\
H_n(B) & H_n(E,F) \arrow[l, "\cong"'] \arrow[r, "\partial"] & H_{n-1}(F) \,.
\end{tikzcd}$$

The isomorphism $\pi_n(E,F) \to \pi_n(B)$ is Lemma 9.7. □

Let us now specialize to the case of the path-loop fibration

$$\Omega X \to PX \to X$$

where X is a simply-connected pointed space. The coefficient system is trivial. Suppose that in fact $\overline{H}_i(X) = 0$ for $i < n$. Since the spectral sequence converges to the homology of a point, we find that $\overline{H}_i(\Omega X) = 0$ for $i < n-1$. The Serre exact sequence, or direct use of the spectral sequence as in the computation of $H_*(\Omega S^n)$, shows this:

Lemma 27.2. *Let X be an $(n-1)$-connected pointed space. The transgression relation provides an isomorphism*

$$\overline{H}_i(X) \to \overline{H}_{i-1}(\Omega X)$$

for $i \leq 2n-2$.

For example, if X is simply connected, we get a commutative diagram

$$\begin{tikzcd}
\pi_2(X) \arrow[r, "\cong"] \arrow[d] & \pi_1(\Omega X) \arrow[d] \\
H_2(X) \arrow[r, "\cong"] & H_1(\Omega X) \,.
\end{tikzcd}$$

Since ΩX is an H-space its fundamental group is abelian, so Poincaré's theorem shows that the Hurewicz homomorphism on the right is an isomorphism. Therefore the map on the left is. This is a case of the Hurewicz theorem! In fact, continuing by induction we discover a proof of the general case of the Hurewicz theorem.

Theorem 27.3 (Hurewicz)**.** *Let $n \geq 1$. Suppose X is a pointed space that is $(n-1)$-connected: $\pi_i(X) = 0$ for $i < n$. Then $\overline{H}_i(X) = 0$ for $i < n$ and the Hurewicz map $\pi_n(X)^{ab} \to H_n(X)$ is an isomorphism.*

Going relative

Any topological concept seems to get more useful if you can extend it to a relative form. So let (B, A) be a pair of spaces. To make the construction for the Serre spectral sequence that we proposed earlier work, we should assume that this is a relative CW complex. Suppose that $E \downarrow B$ is a fibration. The pullback or restriction

$$\begin{tikzcd}
E_A \arrow[r] \arrow[d] & E \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

provides us with a "fibration pair" (E, E_A). Suppose that B is path-connected and A nonempty, pick a basepoint $* \in A$, write F for the fiber of $E \downarrow B$ over $*$ (which is of course also the fiber of $E|_A \downarrow A$ over $*$), and suppose that $\pi_1(B, *)$ acts trivially on $H_*(F)$. With these assumptions, pulling back skelata of B rel A yields the *relative Serre spectral sequence*

$$E^2_{s,t} = H_s(B, A; H_t(F)) \underset{s}{\Longrightarrow} H_{s+t}(E, E_A)\,.$$

Let's apply this right away to prove a relative version of the Hurewicz theorem. We will develop conditions under which

$$h : \pi_i(X, A) \to H_i(X, A)$$

is an isomorphism for all $i \leq n$. We will of course assume that X is path connected and that A is nonempty, which together imply that $H_0(X, A) = 0$. Since $\pi_1(X, A)$ is in general only a pointed set let's begin by assuming that it vanishes. This implies that A is also path connected and that $\pi_1(A) \to \pi_1(X)$ is surjective. The induced map on abelianizations is then also surjective, so by Poincaré's theorem $H_1(A) \to H_1(X)$ is surjective and so $H_1(X, A) = 0$.

Moving up to the next dimension, we may hope that $h : \pi_2(X, A) \to H_2(X, A)$ is then an isomorphism, but $\pi_2(X, A)$ is not necessarily abelian so this can't be right in general. This can be fixed – in fact if we kill the action of $\pi_1(A)$ on $\pi_2(X, A)$ it becomes abelian and the resulting homomorphism to $H_2(X, A)$ is an isomorphism (see [37, Ch. 5, Sec. 7]). But we'll be assuming that $\pi_1(X) = 0$ in a minute anyway, so let's just go ahead now and assume that $\pi_1(A) = 0$. The long exact homotopy sequence then shows that $\pi_2(X, A)$ is a quotient of $\pi_2(X)$ and so is abelian. We'll show that $h : \pi_2(X, A) \to H_2(X, A)$ is then an isomorphism.

We will use the fact (Homework for Lecture 47) that the projection map induces a isomorphism

$$\pi_n(E, E_A) \xrightarrow{\cong} \pi_n(B, A)$$

for any $n \geq 1$. In particular, let F be the homotopy fiber of the inclusion map $A \hookrightarrow X$: that is, the pullback in

$$\begin{tikzcd} F \arrow[r] \arrow[d] & PX \arrow[d] \\ A \arrow[r] & X\,. \end{tikzcd}$$

The path space PX is contractible, so from the long exact homotopy sequence for the pair (PX, F) we find that the maps on the top row of the following commutative diagram are isomorphisms.

$$\begin{tikzcd} \pi_{n-1}(F) \arrow[d, "h"] & \pi_n(PX, F) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "h"] & \pi_n(X, A) \arrow[d, "h"] \\ \overline{H}_{n-1}(F) & H_n(PX, F) \arrow[l, "\cong"'] \arrow[r, "p_*"] & H_n(X, A)\,. \end{tikzcd}$$

Returning to our $n = 2$ case, the left arrow is an isomorphism by Poincaré's theorem, since F is path connected and by our assumptions its fundamental group is abelian. What remains in this case then is to show that homology behaves like homotopy, in the sense that $H_2(PX, F) \to H_2(X, A)$ is an isomorphism.

In general, if we assume that, for some $n \geq 3$, $\pi_i(X, A) = 0$ for $i < n$, then the absolute case of the Hurewicz theorem implies that the left Hurewicz homomorphism is an isomorphism, and we are left wanting to show that $H_n(PX, F) \to H_n(X, A)$ is an isomorphism.

For this we can appeal to the relative Serre spectral sequence for the fibration pair $(PX, F) \downarrow (X, A)$. It takes the form

$$E^2_{s,t} = H_s(X, A; H_t(\Omega X)) \underset{s}{\Longrightarrow} H_{s+t}(PX, F) = \overline{H}_{s+t-1}(F)\,.$$

provided the coefficient system is trivial. Since $H_0(\Omega X) = \mathbb{Z}[\pi_1(X)]$, we are pretty much forced to assume that X is simply connected if we want simple coefficients.

The universal coefficient theorem gives us a handle on the E^2 term:

$$0 \to H_s(X, A) \otimes H_t(\Omega X) \to H_s(X, A; H_t(\Omega X)) \to \operatorname{Tor}(H_{s-1}(X, A), H_t(\Omega X)) \to 0\,.$$

Now is the time to think about using induction on n: This will allow us to use the assumption that $\pi_i(X, A) = 0$ for $i < n-1$ to conclude that $H_i(X, A) = 0$ for $i < n-1$ and that $\pi_{n-1}(X, A) \xrightarrow{\cong} H_{n-1}(X, A)$; but we have the additional assumption that $\pi_{n-1}(X, A) = 0$ as well, so $H_{n-1}(X, A) = 0$ too. The induction begins with the case $n = 2$.

So when $s < n$ both end terms vanish, and the entire spectral sequence is concentrated along and to the right of $s = n$.

We glean two facts from this vanishing result: First, $H_i(PX, F) = 0$ for $i < n$, so $\overline{H}_i(F) = 0$ for $i < n - 1$. We knew this already from the absolute Hurewicz theorem.

The second fact is that $E^2_{n,0}$ survives intact to $E^\infty_{n,0}$: Nothing can hit it, and it can hit nothing. This is also the only nonzero group along the total degree line n, so (using what we know about the bottom edge homomorphism) the projection map induces an isomorphism $H_n(PX, F) \to H_n(X, A)$. This is a spectral sequence "corner argument."

Putting this together:

Theorem 27.4 (Relative Hurewicz theorem)**.** *Let X be a space and A a subspace. Assume both of them are simply connected, and let $n \geq 2$. Assume that $\pi_i(X, A) = 0$ for $2 \leq i < n$. Then $H_i(X, A) = 0$ for $i < n$, and the relative Hurewicz map*

$$\pi_n(X, A) \to H_n(X, A)$$

is an isomorphism.

With more care (see [37, Ch. 5, Sec. 7]) you can avoid the simple connectivity assumption. However, with it in place, you get a converse statement: Suppose that both X and A are simply connected, let $n \geq 2$, and assume that $H_q(X, A) = 0$ for $q < n$. Simple connectivity of X implies that $\pi_1(X, A)$ is trivial, so we have the hypotheses of the relative Hurewicz theorem with $n = 2$, and conclude from $H_2(X, A) = 0$ that $\pi_2(X, A) = 0$. Continuing in this manner, we have the

Corollary 27.5. *Let X be a space and A a subspace. Assume both of them are simply connected, and let $n \geq 2$. Assume that $H_i(X, A) = 0$ for $2 \leq i < n$. Then $\pi_i(X, A) = 0$ for $i < n$, and the relative Hurewicz map*

$$\pi_n(X, A) \to H_n(X, A)$$

is an isomorphism.

By replacing a general map by a relative CW complex, up to weak homotopy, we find the following important corollary (which we state without the simple connectivity assumptions needed to apply our work so far).

Corollary 27.6 (Whitehead theorem)**.** *Let $f : X \to Y$ be a map of path connected spaces and let $n \geq 1$. If $f_* : \pi_q(X) \to \pi_q(Y)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$ then $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$. The converse holds if both X and Y are simply connected.*

Taking $n = \infty$ gives the further corollary:

Corollary 27.7. *Any weak equivalence induces a isomorphism in homology. Conversely, if X and Y are simply connected then any homology isomorphism $f : X \to Y$ is a weak equivalence.*

Combining this with "Whitehead's little theorem," we conclude that if a map between simply connected CW complexes induces an isomorphism in homology then it is a homotopy equivalence.

28 Double complexes and the Dress spectral sequence

A certain very rigid way of constructing a filtered complex occurs quite frequently – and, indeed, the Serre or even the Leray spectral sequence can be constructed in this way. It leads to an easy treatment of the multiplicative properties of the Serre spectral sequence (as well as, in due course, an account of the behavior of Steenrod operations in it).

Double complexes

A *double complex* is a bigraded abelian group $A = A_{*,*}$ together with differentials $d_h : A_{s,t} \to A_{s-1,t}$ and $d_v : A_{s,t} \to A_{s,t-1}$ that commute:

$$d_v d_h = d_h d_v \,.$$

For our purposes we might as well assume that $A_{s,t}$ is "first quadrant":

$$A_{s,t} = 0 \quad \text{unless} \quad s \geq 0 \quad \text{and} \quad t \geq 0 \,.$$

An example is provided by the tensor product of two chain complexes C_* and D_*: define

$$A_{s,t} = C_s \otimes D_t \,, \quad d_h(a \otimes b) = da \otimes b \,, \quad d_v(a \otimes b) = a \otimes db \,.$$

The graded tensor product is then the "total complex," which in general is the chain complex tA given by

$$(tA)_n = \bigoplus_{s+t=n} A_{s,t}$$

with differential determined by sending $a \in A_{s,t}$ to

$$da = d_h a + (-1)^s d_v a \,.$$

Then

$$d^2 a = d(d_h a + (-1)^s d_v a) = (d_h^2 a + (-1)^s d_h d_v a) + (-1)^{s-1}(d_v d_h a + (-1)^s d_v^2 a) = 0 \,.$$

Define a filtration on the chain complex tA as follows:

$$F_p(tA)_n = \bigoplus_{s+t=n,\, s \leq p} A_{s,t} \;\subseteq\; (tA)_n \,.$$

Let's compute the low pages of the resulting spectral sequence. For a start,

$$E^0_{s,t} = \mathrm{gr}_s(tA)_{s+t} = (F_s/F_{s-1})_{s+t} = A_{s,t} \,.$$

The differential in this associated graded object is determined by the vertical differential in A:

$$d^0 a = \pm d_v a \,.$$

Then

$$E^1_{s,t} = H_{s,t}(E^0, d^0) = H_{s,t}(A; d_v) \,,$$

which we might write as $H^v_{s,t}(A)$.

Now d^1 is the part of the differential d that decreases s by 1: for a d_v cycle in $A^{s,t}$,

$$d^1[a] = [d_h a] \,.$$

So

$$E^2_{s,t} = H^h_{s,t}(H^v(A)) \underset{s}{\Longrightarrow} H_{s+t}(tA) \,.$$

But we can do something else as well. A double complex A can be "transposed" to produce a new double complex $A^\top$ with

$$A^\top_{t,s} = A_{s,t}$$

and for $a \in A^\top_{t,s}$

$$d^\top_h(a) = (-1)^s d_v a \quad , \quad d^\top_v(a) = (-1)^t d_h a \,.$$

When I set the signs up like that, then

$$tA^\top \cong tA$$

as complexes. The double complex $A^\top$ has its own filtration and its own spectral sequence,

$${}^\top E^2_{t,s} = H^v_{t,s}(H^h(A)) \underset{t}{\Longrightarrow} H_{s+t}(tA) \,,$$

converging to the same thing.

If $A_{*,*}$ has a compatible multiplication – and we'll let you decide what that means – then the associated spectral sequences are multiplicative, as can easily be seen from the direct construction given in §**??**.

Dress spectral sequence

Andreas Dress [7] (1938–, Bielefeld) developed the following variation of the approach to the Serre spectral sequence originally employed by Serre himself. He proposed to model a general fibration – indeed, a general map – by the product projections

$$\mathrm{pr}_1 : \Delta^s \times \Delta^t \to \Delta^s \,.$$

He used these models to form a "singular" construction associated to any map $\pi : E \to B$.

$$\mathrm{Sin}_{s,t}(\pi) = \left\{ (f,\sigma) : \begin{array}{ccc} \Delta^s \times \Delta^t & \xrightarrow{f} & E \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ \Delta^s & \xrightarrow{\sigma} & B \end{array} \quad \text{commutes} \right\} .$$

Since $\Delta^s \times \Delta^t \downarrow \Delta^s$ is surjective, σ is determined by f. Commutativity says that the map σ is "fiberwise."

This construction sends any map $\pi : E \to B$ to a functor

$$\mathrm{Sin}_{*,*}(\pi) : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathrm{Set}\,,$$

a "bisimplicial set."

Continuing to imitate the construction of singular homology, we will next apply the free R-module functor to this, to get a bisimplicial R-module $R\mathrm{Sin}_{*,*}(\pi)$. The final step is to define boundary maps by taking alternating sums of the face maps. This provides us with a double complex, that I will write $S_{*,*}(\pi)$.

There are two associated spectral sequences. One of them is a singular homology version of the Leray spectral sequence, and specializes to the Serre spectral sequence in case π is a fibration. The other serves to identify what the first one converges to. I will sketch the arguments.

Let's compute the spectral sequence attached to the transposed double complex first. For this, observe that an element of $\mathrm{Sin}_{s,t}(\pi)$ may be regarded as a pair of dotted arrows in the commutative diagram

$$\begin{tikzcd}
\Delta^s \arrow[r, dashed, "\hat{f}"] \arrow[d, dashed, "\sigma"] & E^{\Delta^t} \arrow[d, "\pi"] \\
B \arrow[r, "c"] & B^{\Delta^t}
\end{tikzcd}$$

where c denotes the inclusion of the constant maps. If we form the pullback E'_t in

$$\begin{tikzcd}
E'_t \arrow[r] \arrow[d] & E^{\Delta^t} \arrow[d, "\pi"] \\
B \arrow[r, "c"] & B^{\Delta^t}
\end{tikzcd}$$

this is saying that $\mathrm{Sin}_{s,t}(\pi) = \mathrm{Sin}_s(E'_t)$, so

$$S_{s,t}(\pi) = S_s(E'_t)\,.$$

But the map $E'_t \to E^{\Delta^t}$ is a weak equivalence (because $c : B \to B^{\Delta^t}$ is), so

$$S_*(E'_t) \to S_*(E)$$

is a quasi-isomorphism. This shows that

$${}^{\mathsf{T}}E^1_{s,t} = H_s(E)$$

for every $t \geq 0$.

Now we should think about what the differential in the t direction does. Each face map will induce the identity, so the alternating sums will induce alternately 0 and the identity. The result is that

$${}^{\mathsf{T}}E^2_{s,t} = \begin{cases} H_s(E) & \text{if } t = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

The spectral sequence collapses at this point, and we learn that there is a canonical isomorphism

$$H_*(tS_{*,*}(\pi)) = H_*(E)\,.$$

This is then what the un-transposed spectral sequence will converge to. So how does it begin?

Fix a singular simplex $\sigma : \Delta^s \to B$, and pull $E \downarrow B$ back along it. Any $f : \Delta^s \times \Delta^t \to E$ compatible with σ then factors uniquely as

$$\begin{tikzcd}
\Delta^s \times \Delta^t \arrow[r, dashed] \arrow[rr, bend left, "f"] \arrow[dr, "\mathrm{pr}_1"'] & \sigma^{-1}E \arrow[r] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\
& \Delta^s \arrow[r, "\sigma"] & B
\end{tikzcd}$$

Adjointing this, we find that the set of such f's forms the set of singular t-simplices in a space of sections:

$$\mathrm{Sin}_t\Gamma(\Delta^s, \sigma^{-1}E)\,.$$

Forming the free R-module and then taking the corresponding chain complex gives a chain complex for each $\sigma \in \mathrm{Sin}_s(B)$, namely

$$S_*(\Gamma(\Delta^s, \sigma^{-1}E))\,.$$

So

$$E^1_{s,t} = \bigoplus_{\sigma:\Delta^s\to B} H_t(\Gamma(\Delta^s, \sigma^{-1}E))\,.$$

The association $\sigma \mapsto H_t(\Gamma(\Delta^s, \sigma^{-1}E))$ is a kind of "sheaf," and the E^2-term that results is a kind of sheaf homology of B with these coefficients. This much you can say for a general map π; this is a singular homology form of the Leray spectral sequence.

If π is a fibration, the map $\sigma^{-1}E \downarrow \Delta^s$ is a fibration, and hence trivial because Δ^s is contractible. So the space of sections is then just the space of maps from the base to the fiber. Write F_σ for the fiber over the barycenter of Δ^s, so that

$$\Gamma(\Delta^s, \sigma^{-1}E) \simeq F_\sigma^{\Delta^s} \simeq F_\sigma\,.$$

and

$$E^1_{s,t} \simeq \bigoplus_{\sigma\in\mathrm{Sin}_s(B)} H_t(F_\sigma)\,.$$

The resulting E^2-term is the homology of B with coefficients in a corresponding local coefficient system:

$$E^2_{s,t} = H_s(B; H_t(p^{-1}(-))\,.$$

There are many advantages to this construction. It is transparently natural in the fibration, and a version exists for *any* map.

29 Cohomological spectral sequences

Upper indexing

We have set everything up for homology, but of course there are cohomology versions of everything as well. Given a filtered space

$$\cdots \subseteq F_{-1}X \subseteq F_0X \subseteq F_1X \subseteq \cdots$$

we filtered the singular chains $S_*(X)$ by

$$F_sS_*(x) = S_*(F_sX)\,.$$

Now we will filter the cochains with values in M by

$$F_{-s}S^*(X;M) = \ker(S^*(X;M) \to S^*(F_{s-1}X;M)) .$$

Note the $-s$; this is necessary to produce an *increasing* filtration of $S^*(X;M)$. Note also the $s-1$. This will make the indexing of the multiplicative structure better. For example, most of our filtered spaces will have $F_{-1} = \varnothing$, in which case $F_0S^*(X;M) = S^*(X;M)$ and all the other filtration degrees are subcomplexes of this. In fact, it's standard and convenient to change notation to "upper indexing" as follows:

$$F^s = F_{-s} .$$

Then F^* is a *decreasing* filtration: $F^s \supseteq F^{s+1}$. If $F_{-1}X = \varnothing$, then $F^0S^*(X;M) = S^*(X;M)$.

The singular cochain complex as normally written is the outcome of a similar sign reversal; so the differential is of degree $+1$. The combination of these two reversals produces a spectral sequence with the following "cohomological" indexing:

$$d_r : E_r^{s,t} \to E_r^{s+r,t-r+1} .$$

To set this up slightly more generally, suppose that C^* is a cochain complex equipped with a decreasing filtration F^*C^*. Write

$$\mathrm{gr}^s C^n = F^sC^n/F^{s+1}C^n .$$

Call it *first quadrant* if

- $F^0C^* = C^*$,
- $H^n(\mathrm{gr}^s C^*) = 0$ for $n < s$,
- $\bigcap F^sC^* = 0$.

Filter the cohomology of C^* by

$$F^sH^n(C^*) = \ker(H^n(C^*) \to H^n(F^{s-1}C^*)) .$$

Theorem 29.1. *Let C^* be a cochain complex with a first quadrant decreasing filtration. There is a naturally associated convergent cohomological spectral sequence*

$$E_r^{s,t} \underset{s}{\Longrightarrow} H^{s+t}(C)$$

with

$$E_1^{s,t} = H^{s+t}(\mathrm{gr}^s C^*) .$$

In particular we have the *cohomology Serre spectral sequence of a fibration* $p : E \to B$:

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-)) \underset{s}{\Longrightarrow} H_{s+t}(E) .$$

Product structure

One of the reasons for passing to cohomology is to take advantage of the cup-product. It turns out that the cup product behaves itself in the cohomology Serre spectral sequence of a fibration $p : E \to B$. With a commutative coefficient ring R understood, the local coefficient system $H^*(p^{-1}(-))$ is now a contravariant functor from $\Pi_1(B)$ to graded commutative R-algebras. Such coefficients produce bigraded R-algebra

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-)))$$

that is graded commutative in the sense that

$$yx = (-1)^{|x||y|}xy$$

where $|x|$ and $|y|$ denote total degrees of elements. The entire spectral sequence is then "multiplicative" in the following sense.

- Each $E_r^{*,*}$ is a commutative bigraded R-algebra
- d_r is a derivation: $d_r(xy) = (d_r x)y + (-1)^{|x|}x(d_r y)$.
- The isomorphism $E_{r+1}^{*,*} \cong H^{*,*}(E_r^{*,*})$ is one of bigraded algebras.
- $E_2^{*,*} = H^*(B; H^*(p^{-1}(-)))$ as bigraded R-algebras.
- The filtration on $H^*(E)$ satisfies

  $$F^s H^n(E) \cdot F^{s'} H^{n'}(E) \subseteq F^{s+s'} H^{n+n'}(E)\,,$$

  and the isomorphisms

  $$E_\infty^{s,t} \cong \mathrm{gr}^s H^{s+t}(E)$$

  together form an isomorphism of bigraded R-algebras.

Theorem 29.2. *Let $p : E \to B$ be a Serre fibration, and assume given a commutative coefficient ring R. There is a naturally associated multiplicative cohomological first quadrant spectral sequence of R-modules*

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-)) \underset{s}{\Longrightarrow} H^{s+t}(E)\,.$$

One of the virtues of the construction of the Serre (or more generally Leray) spectral sequence by the method described in Lecture 28 is that the multiplicative structure arises in a natural and explicit way. The bisimplicial set $S_{*,*}(\pi)$ gives rise to a bicosimplicial R-algebra $\mathrm{Map}(S_{*,*}(\pi), R)$, where the R-algebra structure is obtained by simply multiplying in R. Then applying the Alexander-Whitney map in both directions produces a (non-commutative but associative) algebra structure on a double complex, and the resulting filtered complex has the structure of a filtered differential graded algebra. The multiplicative structure of the spectral sequence is then easy to produce, and extends to a description of the effect of Steenrod operations in it as well [36]. The construction from a CW filtration of the base requires us to choose a skeletal approximation of the diagonal. Anyway, I will not make a further attempt to justify the multiplicative behavior of the Serre spectral sequence.

Instead, let's look at an example: The cohomology Gysin sequence for a fibration $p : E \to B$ whose fibers are R-homology $(n-1)$-spheres with compatible R-orientations takes the form

$$\cdots \to H^{s-n}(B) \xrightarrow{\pm e(\xi)\cdot} H^s(B) \xrightarrow{p^*} H^s(E) \xrightarrow{p_*} H^{s-n+1}(B) \to \cdots\,.$$

The identity of the middle map with p^* follows from the edge-homomorphism arguments above but reformulated in cohomology. How about the other two maps?

Euler class

To understand them let's look at the cohomological Serre spectral sequence giving rise to the Gysin exact sequence. It has two nonzero rows, $E_r^{*,0}$ and $E_r^{*,n-1}$. The multiplicative structure provides $E_r^{*,n-1}$ with the structure of a module over $E_r^{*,0}$. The assumed orientation of the spherical fibration determines a distinguished class σ in the R-module $E_2^{0,n-1} = H^0(B; H^{n-1}(F))$ (one that evaluates to 1 on each orientation class – remember, the base may not be connected!), and $E_2^{*,n-1}$ is free as $E_2^{*,0} = H^*(B)$-module on this generator.

The transgression of this element,

$$e = d_n\sigma \in E_n^{n,0} = H^n(B)\,,$$

is a canonically defined class, called the *Euler class* of the R-oriented spherical fibration.

This class determines the entire transgression $H^*(B) \to H^*(B)$ in the Gysin sequence:

$$x \mapsto d_n(x \cdot \sigma) = (-1)^{|x|}xe = \pm ex$$

by the Leibnitz formula, since $d_n x = 0$.

The Euler class is a "characteristic class," in the sense that if we use $f : B' \to B$ to pull the spherical fibration $\xi : E \downarrow B$ back to $f^*\xi : E' \downarrow B'$ (along with the chosen orientation), then

$$f^*(e(\xi)) = e(f^*\xi)\,.$$

In particular E might be the complement of the zero section of an R-oriented real n-plane bundle. The universal case is then $\xi_n : ESO(n) \downarrow BSO(n)$, and we receive a canonical cohomology class

$$e_n = e(\xi_n) \in H^n(BSO(n); R)\,.$$

If we use coefficients in $\mathbb{F}_2$, every n-plane bundle is canonically oriented and we receive a class $e_n \in H^n(BO(n); \mathbb{F}_2)$.

In a sense the Euler class is the fundamental characteristic class: it rules all others. To illustrate its importance, notice that if $p : E \to B$ has a section $s : B \to E$ then the map $p^* : H^*(B) \to H^*(E)$ is a split injection. The Gysin sequence becomes a short exact sequence; $p_* = 0$. Said differently, the edge homomorphism story shows that in that case all differentials hitting the base are trivial; in particular $e(\xi) = 0$. So if $e(\xi) \neq 0$ then the bundle doesn't admit a section. If the bundle was the complement of the zero section in an R-oriented vector bundle, $e(\xi)$ is an obstruction to the existence of a nowhere zero section.

The Euler class gets its name from the following theorem.

Theorem 29.3. *Let M be an R-oriented closed manifold. Then evaluating the Euler class of the tangent bundle τ on the fundamental class of M produces the image in R of the Euler characteristic of M:*

$$< e(\tau), [M] >= \chi(M) \in R\,.$$

Remark 29.4. If B is a finite CW complex of dimension at most the fiber dimension of the vector bundle, then the Euler class is the only obstruction to compressing a classifying map $B \to BSO(n)$ through a map to $BSO(n-1)$, and the Euler class is a complete obstruction to a section. Thus for example the Euler characteristic closed oriented n-manifold vanishes if and only if the manifold admits a nowhere vanishing vector field.

So if M admits a nonvanishing vector field then $\chi(M) = 0$.

Integration along the fiber

How about the last map, $H^s(E) \to H^{s-n+1}(B)$? This is a "wrong-way" or "Umkher" map – it moves in the opposite direction from $p^* : H^s(B) \to H^s(E)$ – and also decreases dimension by the dimension of the fiber. In fact let $p : E \to B$ be any fibration such that $H^*(B; H^t(p^{-1}(-))) = 0$ for all $t \geq n$, and suppose we are given a map of local systems

$$H^n(p^{-1}(-)) \to R$$

to the trivial local system of R-modules. For example the fibers might be closed $(n-1)$-manifolds, equipped with compatible orientations.

Now we have a new edge, an upper edge, and our map is given by a new edge homomorphism:

$$p_* : H^s(E) = F^0 H^s(E) = F^{s-n+1} H^s(E) \twoheadrightarrow E_\infty^{s-n+1,n-1} \hookrightarrow E_2^{s-n+1,n-1} \to H^{s-n+1}(B)\,.$$

This edge homomorphism can sometimes be given geometric meaning as well. With real coefficients, for example, we can use deRham cohomology, and regard the map p_* as "integration along the fiber."

The multiplicative structure of the spectral sequence implies that the Umkher map p_* is a module homomorphism for the graded algebra $H^*(B)$:

$$p_*((p^*x) \cdot y) = x \cdot p_* y\,.$$

This important formula has various names: "Frobenius reciprocity," or the "projection formula."

Loop space of S^n again

Let's try to compute the cup product structure in the cohomology of ΩS^n, again using the Serre spectral sequence for $PS^n \downarrow S^n$. One way to analyze this would be to set up the cohomology version of the Wang sequence, subject of a homework problem. But let's just use the spectral sequence directly. Take $n > 1$.

To begin,

$$E_2^{s,t} = H^s(S^n; H^t(\Omega S^n)) = H^s(S^n) \otimes H^t(\Omega S^n)\,.$$

There are two nonzero columns. Write $\iota_n \in H^n(S^n)$ for the dual of the orientation class. The cohomology transgression $d_n : E_2^{0,n-1} \to E_2^{n,0}$ must be an isomorphism. Write $x \in H^{n-1}(\Omega S^n)$ for the unique class mapping to ι_n.

As in the homology calculation (or because of it) we know that $H^{k(n-1)}(\Omega S^n)$ is an infinite cyclic group. A first question then is: Is the the cup k-th power x^k a generator?

First assume that n is odd, so that $|x| = n-1$ is even. Then by the Leibniz rule

$$d_n x^2 = 2(d_n x)x = 2\iota_n x\,.$$

This is twice the generator of $E_2^{n,n-1}$. In order to kill the generator itself, we must be able to divide x^2 by 2 in $H^{2(n-1)}(\Omega S^n)$. So there is a unique element, call it γ_2, such that $2\gamma_2 = x^2$, and it serves as a generator for the infinite cyclic group $H^{2(n-1)}(\Omega S^n)$.

With this in the bag, let's observe that the transgression of x^k is

$$d_n x^k = k(d_n x)x^{k-1} = k\iota_n x^{k-1}\,.$$

For example

$$d_n x^3 = 3\iota_n x^2 = 3 \cdot 2\iota_n \gamma_2\,.$$

Since $\iota_n\gamma_2$ is a generator of $E_2^{n,2(n-1)}$, the element x^3 must be divisible by $3 \cdot 2 = 3!$: there is a unique element of $H^{3(n-1)}(\Omega S^n)$, call it γ_3, such that $x^3 = 3!\gamma_3$.

This evidently continues: $H^{k(n-1)}(\Omega S^n)$ is generated by a class γ_k such that $x^k = k!\gamma_k$. This implies that these generators satisfy the product formula

$$\gamma_j\gamma_k = (j,k)\gamma_{j+k} \quad , \quad (j,k) = \frac{(j+k)!}{j!k!} \,.$$

This is a *divided power algebra*, denoted by $\Gamma[x]$:

$$H^*(\Omega S^n) = \Gamma[x] \quad \text{for } n \text{ odd}\,, \quad |x| = n-1\,.$$

The answer is the same for any coefficients. With rational coefficient, these divided classes are already present, so

$$H^*(\Omega S^n;\mathbb{Q}) = \mathbb{Q}[x]\,.$$

Then $H^*(\Omega S^n;\mathbb{Z})$, being torsion-free, sits inside this as the sub-algebra generated additively by the classes $x^k/k!$.

Now let's turn to the case in which n is even. Then $|x|$ is odd, so by commutativity $2x^2 = 0$. But $H^{2(n-1)}(\Omega S^n)$ is torsion-free, so $x^2 = 0$.

So we need a new indecomposable element in $H^{2(n-1)}(\Omega S^n)$: Call it y. Choose the sign so that

$$d_n y = \iota_n x \in E_n^{n,n-1}\,.$$

Now $|y| = 2(n-1)$ is even, so

$$d_n y^k = k\iota_n y^{k-1}x$$

and

$$d_n(xy^k) = \iota_n y^k - x \cdot ky^{k-1}\iota_n x = \iota_n y^k$$

(since $x^2 = 0$). Reasoning as before, we find that

$$H^*(\Omega S^n) = E[x] \otimes \Gamma[y] \quad \text{for } n \text{ even}\,, \quad |x| = n-1\,, \quad |y| = 2(n-1)\,,$$

as algebras, again with any coefficients.

30 Serre classes

Let X be a simply connected space. Suppose that $\overline{H}_q(X)$ is a torsion group for all q: every element $x \in H_q(X)$ is killed by some positive integer. This is the same as saying that X has the same rational homology as a point. Is every homotopy group also a torsion group, or can rational homotopy make an appearance? What if the reduced homology was all p-torsion (i.e. every element is killed by some power of p) – must $\pi_*(X)$ also be entirely p-torsion? What if the homology is assumed to be of finite type (finitely generated in every dimension) – must the same be true of homotopy? Serre explained how things like this can be checked, without explicit computation (which is not an option!) by describing what is required of a class $\mathbf{C}$ of abelian groups that allow it to be considered "negligible."

Definition 30.1. A class $\mathbf{C}$ of abelian groups is a *Serre class* if $0 \in \mathbf{C}$, and, for any short exact sequence $0 \to A \to B \to C \to 0$, A and C lie in $\mathbf{C}$ if and only if B does.

Here are some immediate consequences of this definition.

- A Serre class is closed under isomorphisms.
- A Serre class is closed under formation of subgroups and quotient groups.
- Let $A \xrightarrow{i} B \xrightarrow{p} C$ be exact at B. If $A, C \in \mathbf{C}$, then $B \in \mathbf{C}$: In

$$\begin{tikzcd}
 & A \arrow[d] \arrow[dr, "i"] & & & \\
0 \arrow[r] & \ker p \arrow[r] & B \arrow[r] \arrow[drr, "p"'] & \operatorname{coker} i \arrow[r] \arrow[dr] & 0 \\
 & & & & C
\end{tikzcd}$$

  the row is exact and the indicated factorizations exist since $pi = 0$; the surjectivity and injectivity follow from exactness.

Here are the main examples.

Example 30.2. The class of trivial abelian groups; the class $\mathbf{C}_{\mathrm{fin}}$ of all finite abelian groups; the class $\mathbf{C}_{\mathrm{fg}}$ of all finitely generated abelian groups; the class of all abelian groups.

Example 30.3. $\mathbf{C}_{\mathrm{tors}}$, the class of all torsion abelian groups. To see that this is a Serre class, start with a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0\,.$$

It's clear that if B is torsion then so are A and C. Conversely, suppose that A and C are torsion groups. Let $b \in B$. Then $p(nb) = np(b) = 0$ for some $n > 0$, since C is torsion; so there is $a \in A$ such that $i(a) = nb$. But A is torsion too, so $ma = 0$ for some $m > 0$, and hence $mnb = 0$.

Example 30.4. Fix a prime p. The class of p-torsion groups forms a Serre class. More generally, let $\mathcal{P}$ be a set of primes. Define $\mathbf{C}_{\mathcal{P}}$ to be the class of torsion abelian groups A such that if p divides the order of $a \in A$ for some $p \in \mathcal{P}$ then $a = 0$. If $\mathcal{P} = \varnothing$ this is just $\mathbf{C}_{\mathrm{tors}}$. Write $\mathbf{C}_p$ for $\mathbf{C}_{\{p\}}$. This is the class of torsion abelian groups without p-torsion. Since $\mathbb{Z}_{(p)}$ is a direct limit of copies of $\mathbb{Z}$ with bonding maps running through the natural numbers prime to p, $A \in \mathbf{C}_p$ if and only if $A \otimes \mathbb{Z}_{(p)} = 0$. These are the kinds of groups you're willing to ignore if you are only interested in "p-primary" information.

Example 30.5. The intersection of a collection of Serre classes is again a Serre class. For example, $\mathbf{C}_{\mathrm{fin}} \cap \mathbf{C}_p$ is the class of finite abelian groups of order prime to p.

The definition of a Serre class is set up so that it makes sense to work "modulo $\mathbf{C}$." So we'll say that A is "zero mod $\mathbf{C}$" if $A \in \mathbf{C}$. A homomorphism is a "mod $\mathbf{C}$ monomorphism" if its kernel lies in $\mathbf{C}$; a "mod $\mathbf{C}$ epimorphism" if its cokernel lies in $\mathbf{C}$; and a "mod $\mathbf{C}$ isomorphism" if both kernel and cokernel lie in $\mathbf{C}$. So for example $f : A \to B$ is a mod $\mathbf{C}_{\mathrm{tors}}$ isomorphism exactly when $f \otimes 1 : A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is an isomorphism of rational vector spaces.

Lemma 30.6. *Let $\mathbf{C}$ be a Serre class. The classes of mod $\mathbf{C}$ monomorphisms, epimorphisms, and isomorphisms contain all isomorphisms and are closed under composition. The class of mod $\mathbf{C}$ isomorphisms satisfies 2-out-of-3.*

Proof. Form

$$
\begin{tikzcd}
 & & & 0 \arrow[dr] & & & & 0 & & & \\
 & & & & \ker\beta \arrow[rr] \arrow[dr] & & \operatorname{coker}\alpha \arrow[ur] \arrow[ddrr] & & & & \\
 & & & & & B \arrow[ur] \arrow[dr, "\beta"] & & & & & \\
0 \arrow[rr] & & \ker\beta\alpha \arrow[rr] \arrow[uurr] & & A \arrow[rr, "\beta\alpha"] \arrow[ur, "\alpha"] & & C \arrow[rr] \arrow[dr] & & \operatorname{coker}\beta\alpha \arrow[rr] \arrow[dl] & & 0 \\
 & & & \ker\alpha \arrow[ul] \arrow[ur] & & & & \operatorname{coker}\beta \arrow[dl] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 \arrow[ul] & & 0 & & 0 & &
\end{tikzcd}
$$

and check that the outside path is exact. $\square$

Here are some straightforward consequences of the definition:

- Let C_* be a chain complex. If $C_n \in \mathbf{C}$ then $H_n(C_*) \in \mathbf{C}$.
- Suppose F_*A is a filtration on an abelian group. If $A \in \mathbf{C}$, then $\mathrm{gr}_s A \in \mathbf{C}$ for all s. If the filtration is finite (i.e. $F_m = 0$ and $F_n = A$ for some m, n) and $\mathrm{gr}_s A \in \mathbf{C}$ for all s, then $A \in \mathbf{C}$.
- Suppose we have a spectral sequence $\{E^r_{s,t}\}$. If $E^2_{s,t} \in \mathbf{C}$, then $E^r_{s,t} \in \mathbf{C}$ for $r \geq 2$. If $\{E^r\}$ is a first quadrant spectral sequence (so that $E^\infty_{s,t}$ is defined and achieved at a finite stage) it follows that $E^\infty_{s,t} \in \mathbf{C}$. Thus if the spectral sequence comes from a first quadrant filtered complex C and $E^2_{s,t} \in \mathbf{C}$ for all $s + t = n$, then $H_n(C) \in \mathbf{C}$.

The first implication in homology is this: Suppose that $A \subseteq X$ is a pair of path-connected spaces. If two of $\overline{H}_n(A), \overline{H}_n(X), H_n(X, A)$ are zero mod $\mathbf{C}$ for all n, then so is the third. More generally, if you have a ladder of abelian groups (a map of long exact sequences) and two out of every three consecutive rungs are mod $\mathbf{C}$ isomorphisms then so is the third: a mod $\mathbf{C}$ five-lemma.

Serre rings and Serre ideals

To apply this theory to the Serre spectral sequence we need to know that our class is compatible with tensor product. Let's say that a Serre class $\mathbf{C}$ is a *Serre ring* if whenever A and B are in $\mathbf{C}$, $A \otimes B$ and $\mathrm{Tor}(A, B)$ are too. It's a *Serre ideal* if we only require one of A and B to lie in $\mathbf{C}$ to have this conclusion.

All of the examples given above are Serre rings. The ones without finiteness assumptions are Serre ideals.

Here's another closure property we might investigate, and will need. Suppose that $\mathbf{C}$ is a Serre ring and $A \in \mathbf{C}$. Form the classifying space or Eilenberg Mac Lane space $BA = K(A, 1)$. We know that $H_1(K(A, 1)) = A$ (for example by Poincaré's theorem) so it lies in $\mathbf{C}$. How about the higher homology groups? If they are again in $\mathbf{C}$, the Serre ring is *acyclic*.

Acyclicity is a computational issue. Suppose $\mathbf{C} = \mathbf{C}_{\mathrm{fin}}$ for example. By the Künneth theorem (and the fact that $\mathbf{C}_{\mathrm{fin}}$ is a Serre ring), it's enough to consider finite cyclic groups. What is $H_*(BC_n)$, where C_n is a cyclic group of order n? To answer this we can embed C_n into the circle group S^1 as nth roots of unity. The group of complex numbers of norm 1 acts principally on the unit vectors in $\mathbb{C}^\infty$, and that space, S^∞, is contractible. So $\mathbb{C}P^\infty = BS^1$. The subgroup $C_n \subset S^1$ acts principally on this contractible space as well, so

$$BC_n = C_n \backslash S^\infty = (C_n \backslash S^1) \times_{S^1} S^\infty$$

fibers over $\mathbf{C}P^\infty$ with fiber $C_n \backslash S^1 \cong S^1$. Let's study the resulting Serre spectral sequence, first in homology.

In it, $E^2_{s,t} = H_s(\mathbb{C}P^\infty) \otimes H_t(S^1)$. The only possible differential is d^2. The one thing we know about $K(C_n, 1)$ is that is fundamental group is C_n – abelian, so $H_1(K(C_n, 1)) = C_n$. The only way to accomplish this in the spectral sequence is by $d^2 a = n\sigma$, where $\sigma \in H_1(S^1)$ is one of the generators.

This implies that in the cohomology spectral sequence $d_2 e = nx$, where e generates $H^1(S^1)$ and x generates $H^2(\mathbb{C}P^\infty)$. Then the multiplicative structure takes over: $d_2(x^i e) = nx^{i+1}$.

The effect is that $E_3^{s,t} = 0$ for $t > 0$. The edge homomorphism $H^*(\mathbb{C}P^\infty) \to H^*(BC_n)$ is thus surjective, and we find

$$H^*(BC_n) = \mathbb{Z}[x]/(nx)\,, \quad |x| = 2\,.$$

Passing back to homology, we find that $\overline{H}_i(BC_n)$ is cyclic of order n if i is a positive odd integer and zero otherwise. In particular, it is finite, so $\mathbf{C}_{\text{fin}}$ is acyclic.

Since any torsion abelian group A is the direct limit of the directed system of its finite subgroups, we find that $\overline{H}_q(K(A, 1))$ is then torsion as well: so $\mathbf{C}_{\text{tors}}$ is also acyclic.

The calculation also shows that the class of finite p-groups and the class $\mathbf{C}_p$ are acyclic.

To deal with $\mathbf{C}_{\text{fg}}$, we just have to add the infinite cyclic group, whose homology is certainly finitely generated in each degree. So all our examples of Serre rings are in fact acyclic.

Serre classes in the Serre spectral sequence

Let $\mathbf{C}$ be a Serre ideal. If $H_n(X)$ and $H_{n-1}(X)$ are zero mod $\mathbf{C}$ then $H_n(X; M)$ is zero mod $\mathbf{C}$ for any abelian group M, by the universal coefficient theorem. If $\mathbf{C}$ is only a Serre ring, we still reach this conclusion provided $M \in \mathbf{C}$.

The convergence theorem for the Serre spectral sequence shows this:

Proposition 30.7 (Mod $\mathbf{C}$ Vietoris-Begle Theorem). *Let $\pi : E \to B$ be a fibration such that B and the fiber F are path connected, and suppose $\pi_1(B)$ acts trivially on $H_*(F)$. Let $\mathbf{C}$ be a Serre ideal and suppose that $H_t(F) \in \mathbf{C}$ for all $t > 0$. Then $\pi_* : H_n(E) \to H_n(B)$ is a mod $\mathbf{C}$ isomorphism for all n.*

Proof. The universal coefficient theorem guarantees that $E^2_{s,t} = H_s(B; H_t(F)) \in \mathbf{C}$ as long as $t > 0$. The same is thus true of $E^r_{s,t}$ and hence of $E^\infty_{s,t}$, so the edge homomorphism $\pi_* : H_n(E) \to H_n(B)$ is a mod $\mathbf{C}$ isomorphism. □

This theorem admits a refinement that will be useful in proving the mod $\mathbf{C}$ Hurewicz theorem. For one thing, we would like a result that works for a Serre ring, not merely an Serre ideal, in order to cover cases like $\mathbf{C}_{\text{fg}}$

Proposition 30.8. *Let $\pi : E \to B$ be a fibration such that B is simply connected and the fiber F is path connected. Let $\mathbf{C}$ be a Serre ring and suppose that*

- *$H_s(B) \in \mathbf{C}$ for all s with $0 < s < n$, and*

- *$H_t(F) \in \mathbf{C}$ for all $0 < t < n-1$.*

Then $\pi_* : H_i(E, F) \to H_i(B, *)$ is an isomorphism mod $\mathbf{C}$ for all $i \leq n$.

Proof. We appeal to the relative Serre spectral sequence

$$E^2_{s,t} = \overline{H}_s(B; H_t(F)) \underset{s}{\Longrightarrow} H_{s+t}(E, F)\,.$$

At $E^2_{s,t}$, both the $s = 0$ column and the $s = 1$ column vanish. Also, $E^2_{s,t} \in C$ for (s,t) in the rectangle

$$2 \leq s \leq n-1\,, \quad 1 \leq t \leq n-2\,.$$

In total degree i, $i \leq n$, the only group not vanishing mod $\mathbf{C}$ is $E^2_{i,0}$. So the edge homomorphism $\pi_* : H_i(E, F) \to \overline{H}_i(B)$ is a mod $\mathbf{C}$ isomorphism. □

Theorem 30.9 (Mod $\mathbf{C}$ Hurewicz theorem)**.** *Assume that $\mathbf{C}$ is an acyclic Serre ring. Let X be a simply connected space and let $n \geq 2$. Then $\pi_q(X) \in \mathbf{C}$ for all $q < n$ if and only if $\overline{H}_q(X) \in \mathbf{C}$ for all $q < n$, and in that case the Hurewicz map $\pi_n(X) \to H_n(X)$ is a mod $\mathbf{C}$ isomorphism.*

We'll present the proof in the next lecture. For now, a small selection of corollaries:

Corollary 30.10. *Let X be a simply connected space and $n \geq 2$ or $n = \infty$.*
(1) $H_q(X)$ is finitely generated for all $q < n$ if and only if $\pi_q(X)$ is finitely generated for all $q < n$.
(2) Let p be a prime number. $H_q(X)$ is p-torsion for all $q < n$ if and only if $\pi_q(X)$ is p-torsion for all $q < n$.
(3) If $\overline{H}_q(X; \mathbb{Q}) = 0$ for $q < n$, then $\pi_q(X) \otimes \mathbb{Q} = 0$ for $q < n$, and $h : \pi_n(X) \otimes \mathbb{Q} \to H_n(X; \mathbb{Q})$ is an isomorphism.

31 Mod C Hurewicz and Whitehead theorems

Proof of Theorem 30.9. This follows the proof of the Hurewicz theorem, but some extra care is needed. Again we use induction and the path-loop fibration. Again, it will suffice to show that if $\pi_q(X) \in \mathbf{C}$ for $q < n$ then $\pi_n(X) \to H_n(X)$ is an isomorphism – now mod $\mathbf{C}$. To start the induction, with $n = 2$, we can appeal to the Hurewicz isomorphism: the map $\pi_2(X) \to H_2(X)$ is an actual isomorphism.

The inductive step uses the commutative diagram

$$\begin{tikzcd}
\pi_q(X) \arrow[d, "h"] & \pi_q(PX, \Omega X) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "h"] & \pi_{q-1}(\Omega X) \arrow[d, "h"] \\
\overline{H}_q(X) & H_q(PX, \Omega X) \arrow[l] \arrow[r, "\cong"] & H_{q-1}(\Omega X)\,.
\end{tikzcd}$$

Two thing need checking: (1) the map $H_n(PX, \Omega X) \to \overline{H}_n(X)$ is an isomorphism mod $\mathbf{C}$, and (2) the map $h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ is an isomorphism mod $\mathbf{C}$.

Neither of these facts follow from an inductive hypothesis if $\pi_2(X) \neq 0$ (unless $\mathbf{C}$ is the trivial class), but we begin by showing that they do follow from the inductive hypothesis if $\pi_2(X) = 0$.

Suppose $\pi_2(X) = 0$, so that ΩX is simply connected. Since $\pi_i(\Omega X) = \pi_{i+1}(X)$ we know it lies in $\mathbf{C}$ for $i < n-1$. The inductive hypothesis applies to ΩX and shows that $\overline{H}_i(\Omega X) \in \mathbf{C}$ for $i < n-1$ and that $h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ is a mod $\mathbf{C}$ isomorphism. The inductive hypothesis also applies to X of course, and shows that $\overline{H}_i(X) \in \mathbf{C}$ for $i < n$. So we are in position to apply Proposition 30.8 from last lecture to see fact (1).

But if $\pi_2(X) \neq 0$, ΩX is not simply connected. To deal with that, let's take the 2-connected cover in the Whitehead tower: This is a fibration $Y \downarrow X$ with fiber $K = K(\pi_2(X), 1)$. This is where the acyclic condition comes in: since $\pi_2(X) \in \mathbf{C}$, $H_i(K) \in \mathbf{C}$ for $i > 0$. The long exact sequence for the pair (Y, K) shows that

$$\overline{H}_i(Y) \to H_i(Y, K)$$

is a mod $\mathbf{C}$ isomorphism. We will apply Proposition 30.8 to $(Y, K) \downarrow (X, *)$, using the fact that X is simply connected and $H_i(X) \in \mathbf{C}$ for $0 < i < n$. We find that

$$H_i(Y, K) \to H_i(X, *)$$

is a mod $\mathbf{C}$ isomorphism for $i \leq n$. Therefore the projection map $\overline{H}_i(Y) \to \overline{H}_i(X)$ is a mod $\mathbf{C}$ isomorphism for $i \leq n$.

The map $\pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \geq 2$, so our hypothesis applies to Y, and we can perform the inductive step on it instead of on X. $\square$

Corollary 31.1. *Let X be a simply connected space, p a prime, and $n \geq 2$. Then $\pi_i(X) \otimes \mathbb{Z}_{(p)} = 0$ for all $i < n$ if and only if $\overline{H}_i(X; \mathbb{Z}_{(p)}) = 0$ for all $i < n$, and in that case*

$$h : \pi_n(X) \otimes \mathbb{Z}_{(p)} \to H_n(X; \mathbb{Z}_{(p)})$$

is an isomorphism.

Proof. The acyclic Serre ring $\mathbf{C}_p$ consists of abelian groups such that $A \otimes \mathbb{Z}_{(p)} = 0$. $\square$

Now for the relative version!

Theorem 31.2 (Relative mod **C** Hurewicz theorem)**.** *Let $\mathbf{C}$ be an acyclic Serre ideal, and (X, A) a pair of spaces, both simply connected. Fix $n \geq 1$. Then $\pi_i(X, A) \in \mathbf{C}$ for all i with $2 \leq i < n$ if and only if $H_i(X, A) \in \mathbf{C}$ for all i with $2 \leq i < n$, and in that case $h : \pi_n(X, A) \to H_n(X, A)$ is a mod $\mathbf{C}$ isomorphism.*

The proof follows the same line as in the absolute case. But note the requirement here, in the relative case, that $\mathbf{C}$ is a Serre *ideal*. Let me just point out where that assumption is required. We use the same diagram, in which F is the homotopy fiber of the inclusion $A \hookrightarrow X$:

$$\begin{tikzcd}
\pi_{n-1}(F) \arrow[d, "h"] & \pi_n(PX, F) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "h"] & \pi_n(X, A) \arrow[d, "h"] \\
\overline{H}_{n-1}(F) & H_n(PX, F) \arrow[l, "\cong"'] \arrow[r, "p_*"] & H_n(X, A) .
\end{tikzcd}$$

In the proof that p_* is an isomorphism, we'll again use the relative Serre spectral sequence, but now the E^2 term is $E^2_{s,t} = H_s(X, A; H_t(X))$, and we have no control over $H_t(X)$: all our assumptions related to the relative homology.

And this leads on to a mod **C** Whitehead theorem:

Theorem 31.3 (Mod **C** Whitehead theorem)**.** *Let $\mathbf{C}$ be an acyclic Serre ideal, and $f : X \to Y$ a map of simply connected spaces. Fix $n \geq 2$. The following are equivalent.*
(1) $f_* : \pi_i(X) \to \pi_i(Y)$ is a mod $\mathbb{C}$ isomorphism for $i \leq n-1$ and a mod $\mathbf{C}$ epimorphism for $i = n$, and
(2) $f_* : H_i(X) \to H_i(Y)$ is a mod $\mathbb{C}$ isomorphism for $i \leq n-1$ and a mod $\mathbf{C}$ epimorphism for $i = n$.

The theory of Serre classes is quite beautiful, but it does not relate easily to the standard way of working with homology with coefficients. The following lemma forms the link between mod p homology and the mod $\mathbf{C}_p$ Whitehead theorem.

Lemma 31.4. *Let X and Y be spaces whose p-local homology is of finite type, and suppose $f : X \to Y$ induces an isomorphism in mod p homology. Then it induces a mod $\mathbf{C}_p$ isomorphism in integral homology.*

Proof. Since $\mathbb{Z}_{(p)}$ is flat, a homomorphism $f : A \to B$ is a mod $\mathbf{C}$ isomorphism if and only if $f \otimes 1 : A \otimes \mathbb{Z}_{(p)} \to B \otimes \mathbb{Z}_{(p)}$ is an isomorphism.

A finitely generated module over $\mathbb{Z}_{(p)}$ is trivial if it's trivial mod p. So we want to show that the kernel and cokernel of $f_* : H_*(X) \to H_*(Y)$ are trivial after tensoring with $\mathbb{F}_p$.

Form the mapping cone Z of the map f. By assumption it has trivial mod p reduced homology. Since $\mathbb{Z}_{(p)}$ is Noetherian, $H_*(Z; \mathbb{Z}_{(p)})$ is of finite type. The universal coefficient theorem shows that $\overline{H}_*(Z; \mathbb{Z}_{(p)}) \otimes \mathbb{F}_p \hookrightarrow \overline{H}_*(Z; \mathbb{F}_p)$, so we conclude that $\overline{H}_*(Z) \otimes \mathbb{Z}_{(p)} = \overline{H}_*(Z; \mathbb{Z}_{(p)}) = 0$, and hence that $f_* \otimes 1 : H_*(X) \otimes \mathbb{Z}_{(p)} \to H_*(Y) \otimes \mathbb{Z}_{(p)}$ is an isomorphism. □

Corollary 31.5. *Let X and Y be simply connected spaces whose p-local homology is of finite type, and suppose $f : X \to Y$ induces an isomorphism in mod p homology. Then $f_* : \pi_*(X) \otimes \mathbb{Z}_{(p)} \to \pi_*(Y) \otimes \mathbb{Z}_{(p)}$ is an isomorphism.*

This is every topologist's favorite theorem! Absent the fundamental group, you can treat primes one by one.

Some calculations

Let's first compute the homology – well, at least the rational homology – of Eilenberg Mac Lane space $K(A, n)$, for A finitely generated. By the Künneth isomorphism it suffices to do this for A cyclic. When A is any torsion group, the mod C_{tors} Hurewicz theorem shows that $\overline{H}_*(K(A, n); \mathbb{Q}) = 0$. So we will focus on $K(\mathbb{Z}, n)$.

The case $n = 1$ is the circle, whose cohomology is an exterior algebra on one generator of dimension 1: $H^*(K(\mathbb{Z}, 1); \mathbb{Q}) = E[\iota_1]$, $|\iota_1| = 1$.

We know what $H^*(K(\mathbb{Z}, 2); \mathbb{Q})$ is, too, but let's compute it in a way that starts an induction. It also follows the path laid down by Serre in his computation of the mod 2 cohomology of $K(A, n)$, using the fiber sequence

$$K(A, n-1) \to PK(A, n) \to K(A, n) \, .$$

When $n = 2$ there are only two rows – this is a spherical fibration. The class ι_1 must transgress to a generator, call it $\iota_2 \in H^2(K(\mathbb{Z}, 2); \mathbb{Q})$. Proceeding inductively, using $d_2(\iota_2^k \iota_1) = \iota_2^{k+1}$, you find that

$$H^*(K(\mathbb{Z}, 2); \mathbb{Q}) = \mathbb{Q}[\iota_2] \, .$$

When $n = 3$, there is a polynomial algebra in the fiber. Again the fundamental class must transgress to a generator, $\iota_3 = d_3 \iota_2 \in H^3(K(\mathbb{Z}, 3); \mathbb{Q})$. The Leibniz formula gives $d^3(\iota_2^k) = k \iota_3 \iota_2^{k-1}$. This differential is an isomorphism: this is where working over $\mathbb{Q}$ separates from working anywhere else. So we discover that

$$H^*(K(\mathbb{Z}, 3); \mathbb{Q}) = E[\iota_3] \, .$$

This starts the induction, and leads to

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} E[\iota_n] & \text{if } n \text{ is odd} \\ \mathbb{Q}[\iota_n] & \text{if } n \text{ is even} \, . \end{cases}$$

In both cases, the cohomology is free as a graded commutative algebra.

Proposition 31.6. *The homotopy group $\pi_i(S^n)$ is finite for all i except for $i = n$ and if n is even for $i = 2n-1$, when it is finitely generated of rank 1.*

Proof. The case $n = 1$ is special and simple, so suppose $n \geq 2$. Let

$$S^n \to K(\mathbb{Z}, n)$$

represent a generator of $H^n(S^n)$. It induces an isomorphism in π_n and in H_n.

When n is odd, it induces an isomorphism in rational homology, and therefore in rational homotopy.

When n is even, we should compute the cohomology of the fiber F. The class ι_n on the base survives to a generator of $H^n(S^n; \mathbb{Q})$, but ι_n^2 must die. The only way to kill it is by a transgression from a class $\iota_{2n-1} \in H^{2n-1}(F)$: $d_{2n}\iota_{2n-1} = \iota_n^2$. Then the Leibniz formula gives $d_{2n}(\iota_n^k \iota_{2n-1}) = k\iota_n^{k-1}$, leaving precisely the cohomology of S^n. So the fiber has the same rational cohomology as $K(\mathbb{Z}, 2n-1)$. The generator ι_{2n-1} gives a map $F \to K(\mathbb{Z}, 2n-1)$ that induces an isomorphism in rational homology, and hence in rational homotopy.

You might ask: Why couldn't this cancellation happen some other way? You can complete this argument, but perhaps you'll prefer a different approach. Loop the Barratt-Puppe sequence back one notch, to a fiber sequence $K(\mathbb{Z}, n-1) \to F \to S^n$, and work directly in homology. Now $n-1$ is odd, so the entire E^2 term has just four generators. The generator $x \in H_n(S^n)$ must transgress to the fiber (else F would have the wrong homology in dimension $n-1$, or using the relationship between the transgression and the boundary map in homotopy), and what's left at E^{n+1} is just a $\mathbb{Q}$ for $E^2_{0,0}$ and a $\mathbb{Q}$ for $E^2_{n,n-1}$. □

We can identify an element of infinite order in $\pi_{4k-1}(S^{2k})$ in several ways. Here's one. The space $S^m \times S^n$ admits a CW structure with $(m+n-1)$-skeleton given by the wedge $S^m \vee S^n$. There is thus a map

$$\omega : S^{m+n-1} \to S^m \vee S^n$$

that serves as the attaching map for the top cell. Given homotopy classes $\alpha \in \pi_m(X)$ and $\beta \in \pi_n(X)$, we an form the composite

$$S^{m+n-1} \xrightarrow{\omega} S^m \vee S^n \xrightarrow{\alpha \vee \beta} X \vee X \xrightarrow{\nabla} X$$

This defines the *Whitehead product*

$$[-,-] : \pi_m(X) \times \pi_n(X) \to \pi_{m+n-1}(X).$$

When $m = 1$, this is the action of $\pi_1(X)$ on $\pi_n(X)$. Now we can define the *Whitehead square*

$$w_n = [\iota_n, \iota_n] \in \pi_{2n-1}(S^n)\,.$$

When $n = 2k$, it generates an infinite cyclic subgroup.

The same calculation works for a while locally at a prime. Let's look at S^3 for definiteness. Follow the Barratt-Puppe sequence back one stage, to get a fibration sequence

$$K(\mathbb{Z}, 2) \to \tau \geq 4S^3 \to S^3$$

In the spectral sequence, with integral coefficients,

$$E_2^{*,*} = E[\sigma] \otimes \mathbb{Z}[\iota_2]\,.$$

The class ι_2 must transgress to σ (at least up to sign), and then

$$d_2(\iota_2^k) = k\sigma\iota_2^{k-1}\,.$$

This map is always injective, leaving

$$E_3^{3,2k-2} = \mathbb{Z}/k\mathbb{Z}$$

and nothing else except for $E_3^{0,0} = \mathbb{Z}$. The result is that

$$H_{2k}(\tau_{\geq 4}S^3) = \mathbb{Z}/k\mathbb{Z}\,, \quad k \geq 1\,.$$

The first time p-torsion appears is in dimension $2p$: $H_{2p}(\tau_{\geq 4}S^3) = \mathbb{Z}/p\mathbb{Z}$. This is the mod $\mathbf{C}_p$ Hurewicz dimension, so $\pi_i(S^3)$ has no p-torsion in dimension less than $2p$, and $\pi_{2p}(S^3)\otimes\mathbb{Z}_{(p)} = \mathbb{Z}/p\mathbb{Z}$.

32 Freudenthal, James, and Bousfield

Suspension

The transgression takes on a particularly simple form if the total space is contractible.

Remember the adjoint pair

$$\Sigma : \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : \Omega\,.$$

The adjunction morphisms

$$\sigma : X \to \Omega\Sigma X\,, \quad \mathrm{ev} : \Sigma\Omega X \to X$$

are given by

$$\sigma(x)(t) = [x,t]\,, \quad \mathrm{ev}(\omega, t) = \omega(t)\,.$$

Proposition 32.1. *Let X be a path connected space. The transgression relation*

$$\overline{H}_n(X) \rightharpoonup \overline{H}_{n-1}(\Omega X)$$

associated to the path loop fibration $p : PX \to X$ is the converse of the relation defined by the map

$$\overline{H}_{n-1}(\Omega X) = \overline{H}_n(\Sigma\Omega X) \xrightarrow{\mathrm{ev}_*} \overline{H}_n(X)\,.$$

Proof. Recall that the transgression relation is given (in this case) by the span

$$\begin{tikzcd} & H_n(PX, \Omega X) \arrow[dl, "p_*"'] \arrow[dr, "\partial"] & \\ \overline{H}_n(X) & & \overline{H}_{n-1}(\Omega X)\,. \end{tikzcd}$$

It consists of the subgroup

$$\{(x,y) \in \overline{H}_n(X) \times \overline{H}_{n-1}(\Omega X) : \exists\; z \in H_n(PX,\Omega X) \text{ such that } p_*z = x \text{ and } \partial z = y\}\,.$$

We are claiming that this is the same as the subgroup

$$\{(x,y) \in \overline{H}_n(X) \times \overline{H}_{n-1}(\Omega X) : \exists\; w \in \overline{H}_n(\Sigma\Omega X) \text{ such that } \mathrm{ev}_*w = x \text{ and } iw = y\}$$

determined by the span

$$\begin{tikzcd} & \overline{H}_n(\Sigma\Omega X) \arrow[dl, "\mathrm{ev}_*"'] \arrow[dr, "i"] & \\ \overline{H}_n(X) & & \overline{H}_{n-1}(\Omega X) \end{tikzcd}$$

where $i : \overline{H}_{n-1}(\Omega X) \xrightarrow{\cong} \overline{H}_n(\Sigma\Omega X)$ is the canonical isomorphism.

To see this, we just have to remember how the boundary map and the isomorphism i are related. This is a general point. So suppose we have a space X and a subspace A, so we are interested in $i : \overline{H}_n(\Sigma X) \to \overline{H}_{n-1}(X)$ and the the boundary map $\partial : H_n(X, A) \to \overline{H}_{n-1}(A)$. The latter may be described geometrically in the following way. Form the mapping cylinder M of the inclusion map $A \to X$. Then $A \hookrightarrow M$ is a cofibration with cofiber ΣA, and we have the span

$$\begin{tikzcd} & (M, A) \arrow[dl] \arrow[dr] & \\ (X, A) & & (\Sigma A, *) \end{tikzcd}$$

in which the left arrow is a homology isomorphism. The boundary map is induced by this span, together with the isomorphism i.

Specializing to the pair $(PX, \Omega X)$ gives commutativity of part of the diagram

$$\begin{tikzcd} & & H_n(M, \Omega X) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\ & H_n(PX, \Omega X) \arrow[dl, "p_*"'] \arrow[drrr, "\partial"' near end] & & H_n(\Sigma\Omega X, *) \arrow[dlll, "\mathrm{ev}_*" near end] \arrow[dr, "i"] & \\ \overline{H}_n(X) & & & & \overline{H}_{n-1}(\Omega X)\,. \end{tikzcd}$$

The other part follows from homotopy commutativity of

$$\begin{tikzcd} (M, \Omega X) \arrow[r] \arrow[d] & (\Sigma\Omega X, *) \arrow[d, "\mathrm{ev}"] \\ (PX, \Omega X) \arrow[r, "p"] & (X, *) \end{tikzcd} \qquad \begin{tikzcd} \sigma; (\omega, t) \arrow[rr, mapsto] \arrow[d, mapsto] & & *; [\omega, t] \arrow[d, mapsto] \\ \sigma(0); \omega \arrow[r, mapsto] & \sigma(0); * & *; \omega(t)\,. \end{tikzcd}$$

Notation: the first entry is the map on $PX = \{\sigma : I \to X \text{ such that } \sigma(0) = *\}$; the second entry is the map on $\Omega X \times I$. A homotopy between the two branches is given at time s by

$$\sigma; \omega \mapsto \sigma(s); (t \mapsto \omega(st))\,.$$

This diagram shows that the two relations are identical. $\square$

The evaluation map $\Sigma\Omega X \to X$ also admits an interesting interpretation in cohomology, with coefficients in an abelian group π:

$$\begin{tikzcd}
\overline{H}^n(X;\pi) \arrow[r] \arrow[dd, "\cong"] & \overline{H}^{n-1}(\Omega X;\pi) \arrow[d, "\cong"] \\
 & {[\Omega X, K(\pi,n-1)]_*} \arrow[d, "\cong"] \\
{[X, K(\pi,n)]_*} \arrow[r, "\Omega"] & {[\Omega X, \Omega K(\pi,n)]_*}
\end{tikzcd}$$

commutes.

Our identification of the evaluation map as the converse of a transgression allows us to invoke the Serre exact sequence. After all, if the total space is contractible, every third term in the Serre exact sequence vanishes, and the remaining map, the transgression, is an isomorphism. In fact, in that case we get just a little extra, the last clause in the following proposition, which we state in the generality of working modulo a Serre ring.

Proposition 32.2. *Let* $\mathbf{C}$ *be a Serre ring. Let* $n \geq 1$ *and suppose* X *is simply connected and that* $\overline{H}_i(X) \in \mathbf{C}$ *for all* $i < n$*. Then the evaluation map* $ev_* : \overline{H}_{i-1}(\Omega X) \to \overline{H}_i(X)$ *is an isomorphism mod* $\mathbf{C}$ *for* $i < 2n-1$ *and an epimorphism mod* $\mathbf{C}$ *for* $i = 2n-1$*.*

This result leads the way to the "suspension theorem" of Hans Freudenthal (1905–1990; German, working in Amsterdam, escaped from a labor camp during World War II). The relevant adjunction morphism is now the "suspension"

$$\sigma_X : X \to \Omega\Sigma X\,.$$

The formalism of adjunction guarantees commutativity of

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma\sigma_X"] \arrow[dr, "1"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}_{\Sigma X}"] \\
 & \Sigma X\,,
\end{tikzcd}$$

which shows for a start that σ_X induces a split monomorphism in reduced homology. But we also know from 32.2 that if X is $(n-1)$ connected, the evaluation map in

$$\begin{tikzcd}
H_i(X) \arrow[r, "(\sigma_X)_*"] \arrow[dr, "="'] & H_i(\Omega\Sigma X) \arrow[d, "(\mathrm{ev}_{\Sigma X})_*"] \\
 & H_{i+1}(\Sigma X)
\end{tikzcd}$$

is an isomorphism mod $\mathbf{C}$ for $i < 2n$: so the same is true for $(\sigma_X)_*$. Now we can apply the mod $\mathbf{C}$ Whitehead theorem to conclude:

Theorem 32.3 (Mod $\mathbf{C}$ Freudenthal suspension theorem)**.** *Let* $\mathbf{C}$ *be an acyclic Serre ideal and* $n \geq 1$*. Let* X *be a simply connected space such that* $\overline{H}_i(X)$ *is zero mod* $\mathbf{C}$ *for* $i < n$*. Then the suspension map*

$$\pi_i(X) \to \pi_i(\Omega\Sigma X) = \pi_{i+1}(\Sigma X)$$

is a mod $\mathbf{C}$ *isomorphism for* $i < 2n-1$ *and a mod* $\mathbf{C}$ *epimorphism for* $i = 2n-1$*.*

Corollary 32.4. *Let $n \geq 2$. The suspension map*

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n-1$ and an epimorphism for $i = 2n-1$.

For example, $\pi_2(S^2) \to \pi_3(S^3)$ is an isomorphism (the degree is a stable invariant), while $\pi_3(S^2) \to \pi_4(S^3)$ is only an epimorphism: the Hopf map $S^3 \to S^2$ suspends to a generator of $\pi_4(S^3)$, which as we saw has order 2.

In any case, the Freudenthal suspension theorem show that the sequence

$$\pi_k(X) \to \cdots \to \pi_{n+k}(\Sigma^n X) \to \pi_{n+1+k}(\Sigma^{n+1} X) \to \cdots$$

stabilizes. The direct limit is the *reduced kth stable homotopy group* of the pointed space X, $\pi_k^s(X)$. These functors turn out to form a generalized homology theory. The coefficients form a graded commutative ring, the *stable homotopy ring*

$$\pi_*^s = \pi_*^s(S^0) = \lim_{n\to\infty} \pi_{*+n}(S^n)\,.$$

EHP sequence

The homotopy groups of spheres are related to each other via the suspension maps, but it turns out that there is more, based on the following theorem of Ioan James. (Ioan Mackenzie James (1928–) worked mainly at Oxford.)

Proposition 32.5. *Let $n \geq 2$. There is a map $h : \Omega S^n \to \Omega S^{2n-1}$ that induces an isomorphism in $H_{2n-2}(-)$.*

Granting this, we can compute the entire effect in cohomology. When n is even, $n = 2k$, the generator $y \in H^{4k-2}(\Omega S^{4k-1})$ hits the divided power generator in $H^{4k-2}(\Omega S^{2k})$, and hence embeds $H^*(\Omega S^{4k-1})$ into $H^*(\Omega S^{2k})$ isomorphically in dimensions divisible by $(4k-2)$. The induced map in homology thus has the same behavior. It follows that the homotopy fiber has the homology of S^{2k-1}. But the suspension map $S^{2k-1} \to \Omega S^{2k}$ certainly composes into ΩS^{4k-1} to a null map, and hence lifts to a map to the homotopy fiber inducing a homology isomorphism. By Whitehead's theorem, it is a weak equivalence.

The Whitehead square $w_{2k} : S^{4k-1} \to S^{2k}$ has the property that the composite

$$\Omega S^{4k-1} \xrightarrow{\Omega w_{2k}} \Omega S^{2k} \xrightarrow{h} \Omega S^{4k-1}$$

is an isomorphism in homology away from 2. So, using the multiplication in ΩS^{2k}, there is a map,

$$S^{2k-1} \times \Omega S^{4k-1} \to \Omega S^{2k}$$

that induces an isomorphism in homology away from 2, and hence by the mod $\mathbf{C}$ Whitehead theorem in homotopy away from 2. For this reason, even spheres are not very interesting homotopy theoretically away from 2.

When n is odd, $y \in H^{2n-2}(\Omega S^{2n-1})$ maps to the divided square of $x \in H^{n-1}(\Omega S^n)$. This implies that

$$\gamma_k(y) = \frac{y^k}{k!} \mapsto \frac{(x/2)^k}{k!} = \frac{(2k)!}{2^k k!}\gamma_{2k}(x)\,.$$

A little thought shows that the fraction here is odd, so the map is still an isomorphism in $\mathbb{Z}_{(2)}$ homology in dimensions divisible by $2n-2$, and hence the fiber has the 2-local homology of S^{n-1}. The mod $\mathbf{C}_2$ Whitehead theorem shows that it also has the 2-local homotopy of S^{n-1}. We conclude:

Theorem 32.6. *For any positive even integer n there is a fiber sequence*

$$S^{n-1} \to \Omega S^n \to \Omega S^{2n-1}\,.$$

Localized at 2, this sequence exists for n odd as well.

The long exact homotopy sequence then gives us the *EHP sequence*

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "p"'] \\
\pi_k(S^{n-1}) \arrow[r, "e"] & \pi_{k+1}(S^n) \arrow[r, "h"] & \pi_{k+1}(S^{2n-1}) \arrow[dll, "p"'] \\
\pi_{k-1}(S^{n-1}) \arrow[r, "e"] & \pi_k(S^n) \arrow[r, "h"] & \pi_k(S^{2n-1}) \arrow[dll, "p"'] \\
\cdots & &
\end{tikzcd}$$

of homotopy groups (localized at 2 if n is odd).

These sequences link together to form an exact couple! You can see this clearly from the diagram of fiber sequences obtained by looping down the sequences of Theorem 32.6. Locally at 2, we have a diagram in which each L is a fiber sequence.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{s-1}S^{s-1} \arrow[r] \arrow[d] & \Omega^s S^s \arrow[r] \arrow[d] & \Omega^{s+1}S^{s+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^\infty S^\infty \\
 & \Omega^{s-1}S^{2s-3} & \Omega^s S^{2s-1} & \Omega^{s+1}S^{2s+1} & &
\end{tikzcd}$$

The limiting space $\Omega^\infty S^\infty$ has homotopy equal to π_*^s.

The resulting spectral sequence, the *EHP spectral sequence*, has the form

$$E^1_{s,t} = \pi_{2s+1+t}(S^{2s+1}) \underset{s}{\Longrightarrow} \pi^s_{s+t}\,.$$

Here's a picture, taken from [23]. In it, 2^3 represents the elementary abelian group of order 8 and "∞" means an infinite cyclic group.

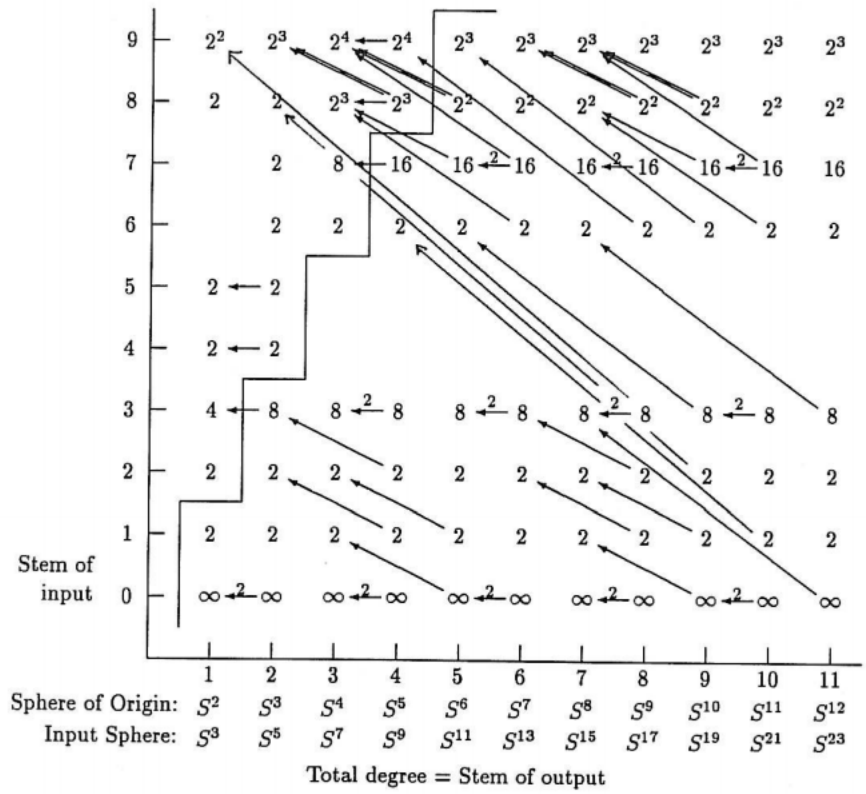


FIGURE 3. The EHP spectral sequence

Bousfield localization

I can't leave the subject of Serre classes without mentioning a more recent and more geometric approach to localization in algebraic topology, due to Bousfield (following diverse early ideas of Dennis Sullivan, Mike Artin and Barry Mazur, and Frank Adams). (A. K. ("Pete") Bousfield was a student of Dan Kan's at MIT.)

Theorem 32.7 (Bousfield, [3]). *Let E_* be any generalized homology theory and X any CW complex. There is a space $L_E X$ and a map $X \to L_E X$ that is terminal in the homotopy category among E_*-equivalences from X.*

So $L_E X$ is as far away (to the right) from X as possible while still receiving an E_*-equivalence from it. The localization strips away all features not detected by E-homology.

The class of maps given by E_*-equivalences determines a class of objects: A space W is *E_*-local* if for every E_*-equivalence $X \to Y$ between CW complexes the induced map $[X, W] \leftarrow [Y, W]$ is bijective. You can't tell two E_*-equivalent spaces apart by mapping them into an E_*-local space.

Theorem 32.8 (Addendum to Theorem 32.7). *For any CW complex X, $L_E X$ is E_*-local, and the localization map $X \to L_E X$ is initial among maps to E_*-local spaces.*

The functor L_E is "Bousfield localization" at the homology theory E_*. The subcategory of E_*-local spaces affords the ultimate extension of the Whitehead theorem:

Lemma 32.9. *Any E_*-equivalence $f : X \to Y$ between E_*-local CW complexes is a homotopy equivalence.*

Proof. Take $W = X$ in the definition of "E_*-local": then the identity map $X \to X$ lifts in the homotopy category uniquely through a map $g : Y \to X$. By construction $gf = 1_X$. But then both fg and 1_Y lift $f : X \to Y$ across f, and hence must be equal by uniqueness. □

So the Whitehead theorem can be phrased as saying that any simply connected CW complex is $H\mathbb{Z}_*$-local.

Another example is given by rational homology $H\mathbb{Q}_*$.

Proposition 32.10. *A simply connected CW complex is $H\mathbb{Q}_*$-local if and only if its homology in each positive dimension is a rational vector space.*

In this case we can also compute the homotopy: For a simply connected CW complex X, $\pi_*(X) \to \pi_*(L_{H\mathbb{Q}}X)$ simply tensors the homotopy with $\mathbb{Q}$. This is the beginning of an extensive development of "rational homotopy theory," pioneered independently by Daniel Quillen and Dennis Sullivan. The entire homotopy theory of simply connected rational spaces of finite type over $\mathbb{Q}$ is equivalent to the opposite of the homotopy theory of commutative differential graded $\mathbb{Q}$-algebras that are simply connected and of finite type. The quest for analogous completely algebraic descriptions of other sectors of homotopy theory has been a major research objective over the past half century.

Bousfield localization at $H\mathbb{F}_p$ is trickier, because the map from S^n to the Moore space M with homology given by the p-adic integers $\mathbb{Z}_p$ in dimension n is an isomorphism in mod p homology. In fact $L_{H\mathbb{F}_p}S^n = M$: so in this case Bousfield localization behaves like a completion.

When the fundamental group is nontrivial, even localization at $H\mathbb{Z}$ can lead to unexpected results. For example, let Σ_∞ be the group of permutations of a countably infinite set that move only finitely many elements. Then

$$L_{H\mathbb{Z}}B\Sigma_\infty \simeq \Omega_0^\infty S^\infty \,.$$

a single component of the union of $\Omega^s S^s$'s. This is the "Barratt-Priddy-Quillen theorem."

For another example, let R be a ring and $GL_\infty(R)$ the increasing union of the groups $GL_n(R)$. The homotopy groups of the space $L_{H\mathbb{Z}}BGL_\infty(R)$ formed Quillen's first definition of the higher algebraic K-theory of R.

Bibliography

[1] Michael Atiyah, Thom complexes, Proceedings of the London Philosophical Society **11** (1961) 291–310.

[2] Michael Atiyah, Bordism and cobordism, Proceedings of the Cambridge Philosophical Society **57** (1961) 200–208.

[3] Aldridge Knight Bousfield, The localization of spaces with respect to homology, Topology **14** (1975) 133–150.

[4] Glen Bredon, *Topology and Geometry*, Graduate Texts in Mathematics **139**, Springer-Verlag, 1993.

[5] Edgar Brown, Cohomology Theories, Annals of Mathematics **75** (1962) 467–484.

[6] Albrecht Dold, *Lectures on Algebraic Topology*, Springer-Verlag, 1995.

[7] Andreas Dress, Zur Spektralsequenz einer Faserung, Inventiones Mathematicae **3** (1967) 172-178.

[8] Björn Dundas, *A Short Course in Differential Topology*, Cambridge Mathematical Textbooks, 2018.

[9] Rudolph Fritsch and Renzo Piccinini, *Cellular Structures in Topology*, Cambridge University Press, 1990.

[10] Martin Frankland, Math 527 - Homotopy Theory: Additional notes, `http://uregina.ca/~franklam/Math527/Math527_0204.pdf`

[11] Paul Goerss and Rick Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics **174**, Springer-Verlag, 1999.

[12] Alan Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[13] Dale Husemoller, *Fiber Bundles*, Graduate Texts in Mathematics **20**, Springer-Verlag, 1993.

[14] Sören Illman, The equivariant triangulation theorem for actions of compact Lie groups. Mathematische Annalen **262** (1983) 487–501.

[15] Niles Johnson, Hopf fibration – fibers and base, `https://www.youtube.com/watch?v=AKotMPGFJYk`.

[16] Dan Kan, Adjoint funtors, Transactions of the American Mathematical Society **87** (1958) 294–329.

[17] Anthony Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics 140, Birkaüser, 2002.

[18] Wolfgang Lück, Survey on classifying spaces for families of subgroups, `https://arxiv.org/abs/math/0312378`.

[19] Saunders Mac Lane, *Homology*, Springer Verlag, 1967.

[20] Saunders Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5**, Springer-Verlag, 1998.

[21] Jon Peter May, *A Consise Course in Algebraic Topology*, University of Chicago Press, 1999, `https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`.

[22] Haynes Miller, Leray in Oflag XVIIA: The origins of sheaf theory, sheaf cohomology, and spectral sequences, *Jean Leray (1906–1998)*, Gazette des Mathematiciens **84** suppl (2000) 17–34. `http://math.mit.edu/~hrm/papers/ss.pdf`

[23] Haynes Miller and Douglas Ravenel, Mark Mahowald's work on the homotopy groups of spheres, *Algebraic Topology, Oaxtepec 1991*, Contemporary Mathematics **146** (1993) 1–30.

[24] John Milnor, The geometric realization of a semi-simplicial complex, Annals of Mathematics **65** (1957) 357–362.

[25] John Milnor, The Steenrod algebra and its dual, Annals of Mathematics **67** (1958) 150–171.

[26] John Milnor, A procedure for killing homotopy groups of differentiable manifolds, Proceedings of Symposia in Pure Mathematics, **III** (1961) 39–55.

[27] John Milnor and Jim Stasheff, *Fiber Bundles*, Annals of Mathematics Studies **76**, 1974.

[28] Steve Mitchell, Notes on principal bundles and classifying spaces.

[29] Steve Mitchell, Notes on Serre Fibrations.

[30] James Munkres, *Topology*, Prentice-Hall, 2000.

[31] Daniel Quillen, *Homotopical Algebra*, Springer Lecture Notes in Mathematics **43**, 1967.

[32] *Hommes de Science: 28 portraits*, Hermann, 1990.

[33] Graeme Segal, Classifying spaces and spectral sequences, Publications mathématiques de l'IHES 34 (1968) 105–112.

[34] Paul Selick, *Introduction to Homotopy Theory*, American Mathematical Society, 1997.

[35] Jean-Pierre Serre, Homologie singulière des espaces fibrés. Applications. Annals of Mathematics **54** (1951), 425–505.

[36] William Singer, Steenrod squares in spectral sequences. I, II. Transactions of the Amererican Mathematical Society **175** (1973) 327–336 and 337–353.

[37] Edwin Spanier, *Algebraic Topology*, McGraw Hill, 1966, and later reprints.

[38] Robert Stong, *Notes on Cobordism Theory*, Mathematical Notes, Princeton University Press, 1968.

[39] Neil Strickland, The category of CGWH spaces, `http://www.neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`.

[40] Arne Strøm, A note on cofibrations, Mathematica Scandinavica **19** (1966) 11–14.

[41] René Thom, Quelques propriétés globales des variétés différentiables, Commentarii Mathematici Helvitici **28** (1954) 17–86.

[42] Tammo tom Dieck, *Algebraic Topology*, European Mathematical Society, 2008.

[43] Kalathoor Varadarajan, *The finiteness obstruction of C. T. C. Wall*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley and Sons, 1989.

[44] Charles Terrence Clegg Wall, Finiteness conditions for CW-complexes, Annals of Mathematics **81** (1965) 56–69.

[45] Charles Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics **38**, 1994.

MIT OpenCourseWare
https://ocw.mit.edu

18.906 Algebraic Topology II
Spring 2020

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.